Lattice QCD and QCD phase diagram

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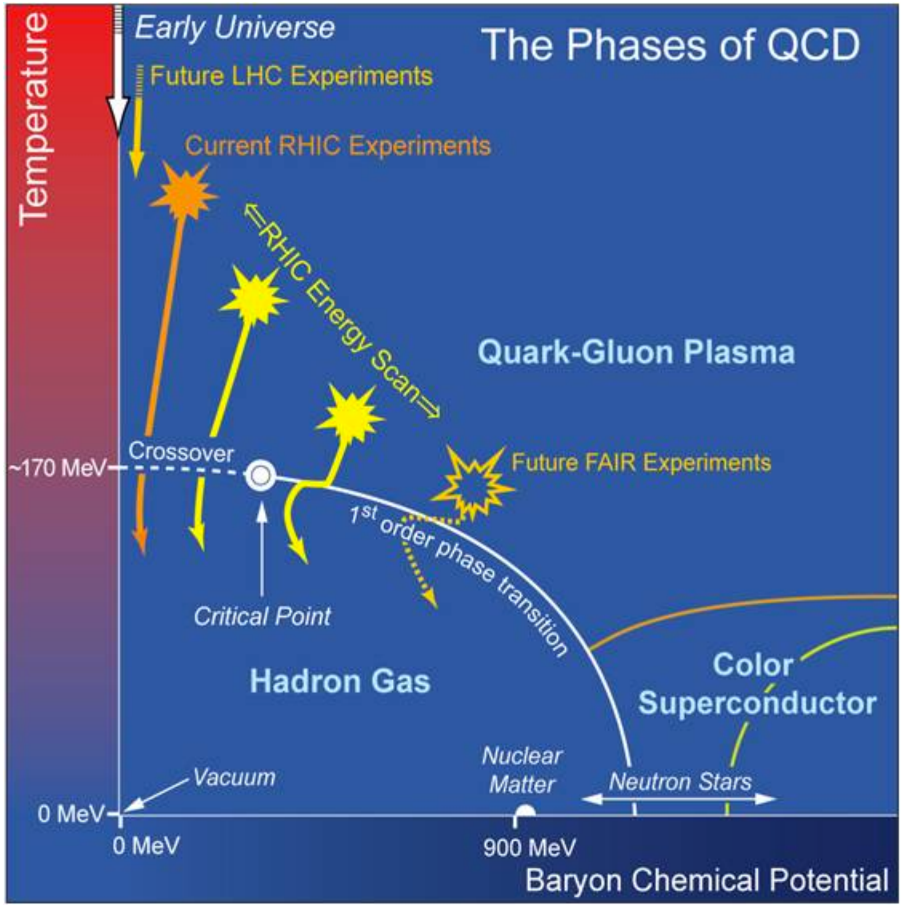
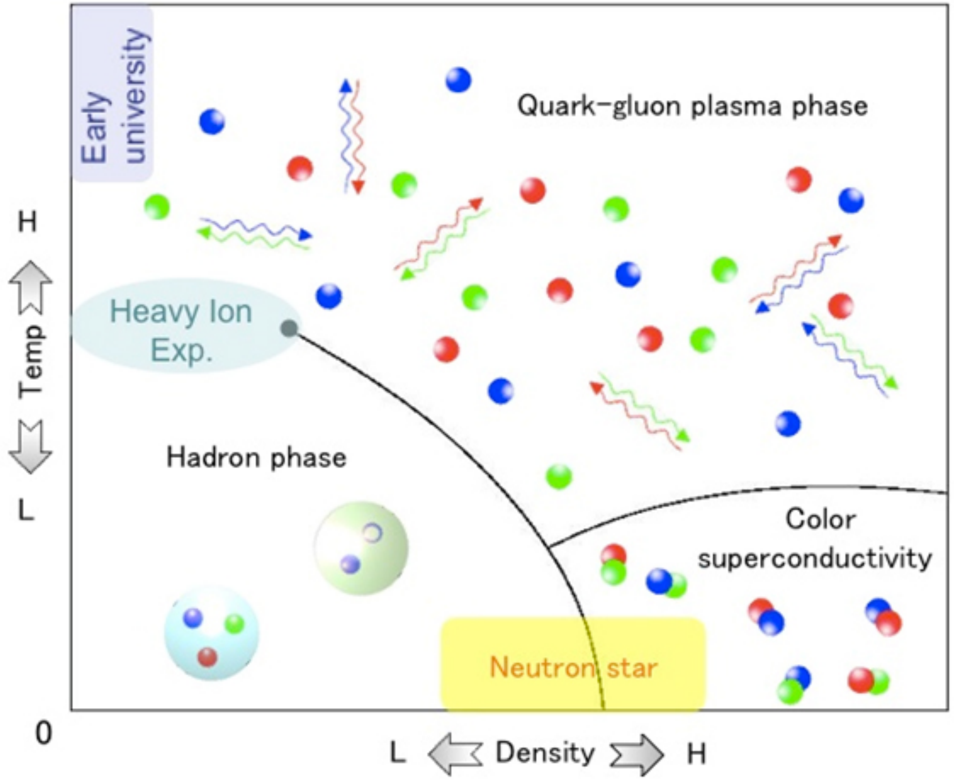
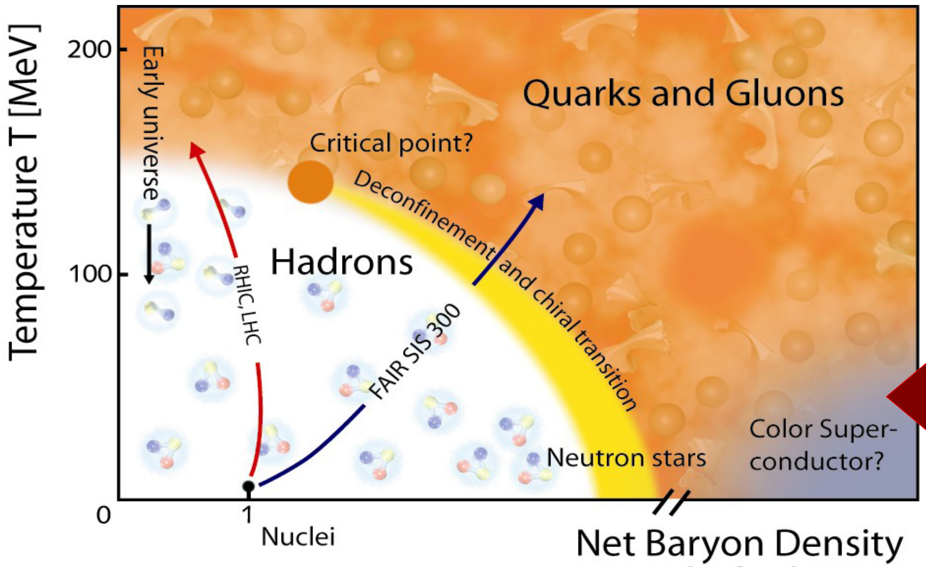
2. QCD at extreme conditions

QCD has complicated phase structure as a function of temperature and density

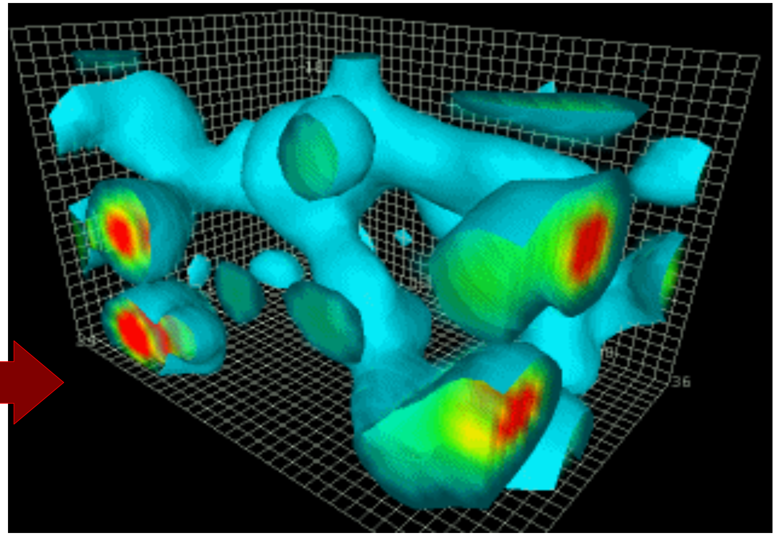




Why are heavy-ion collision experiments special for QCD?



Phase structure of QCD



Nontrivial QCD vacuum

Color symmetry

C/P/T-invariance

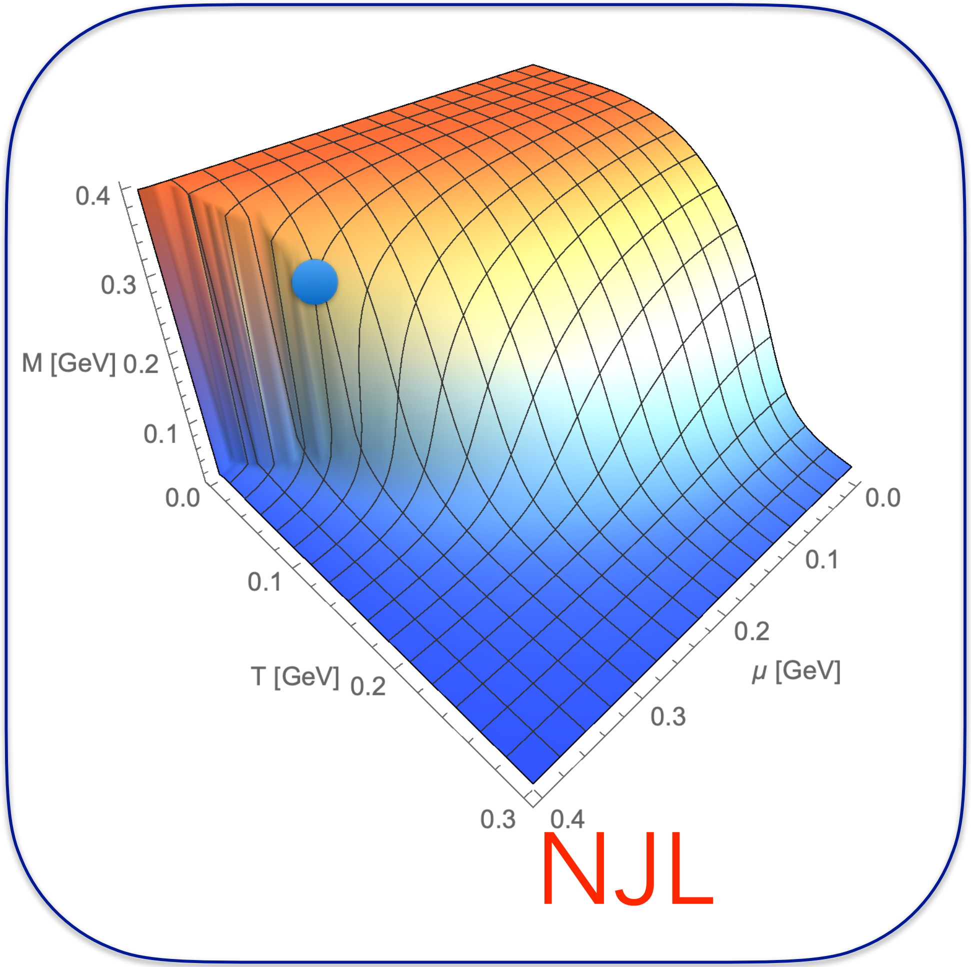
Z(N) center symmetry

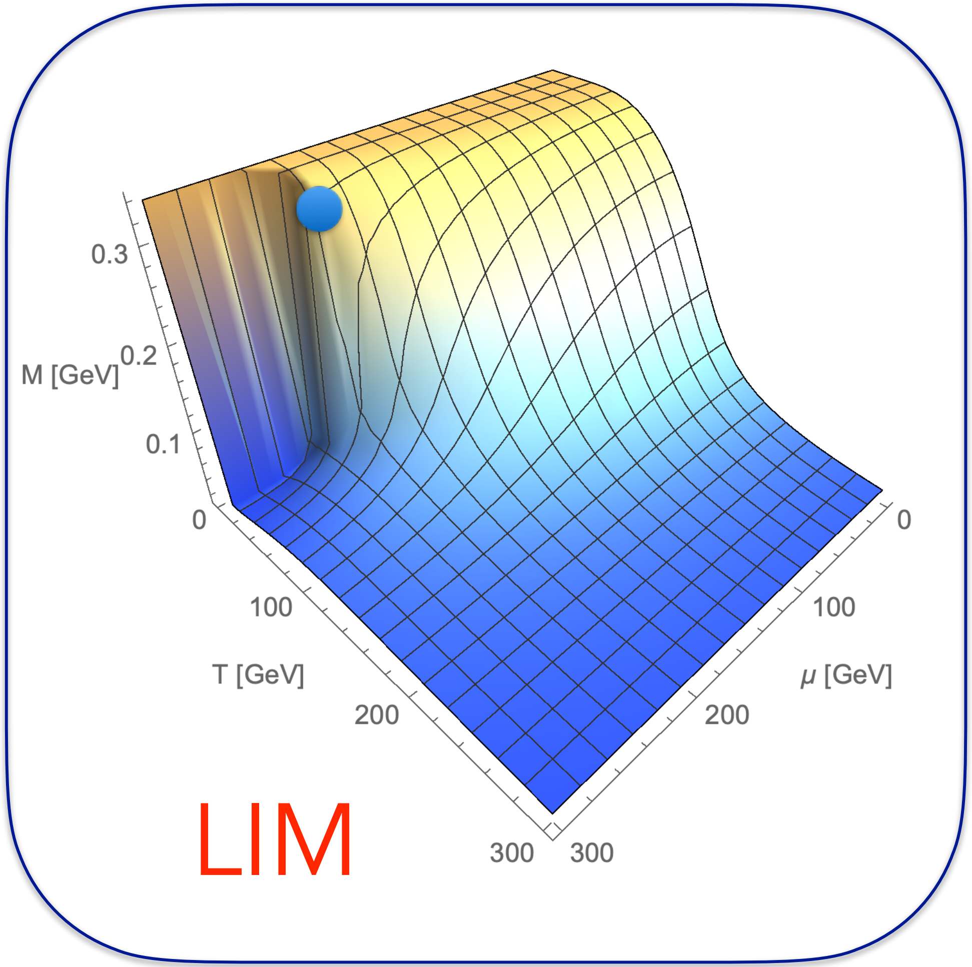
Chiral symmetry

Flavor symmetry

Symmetries and their breakdown

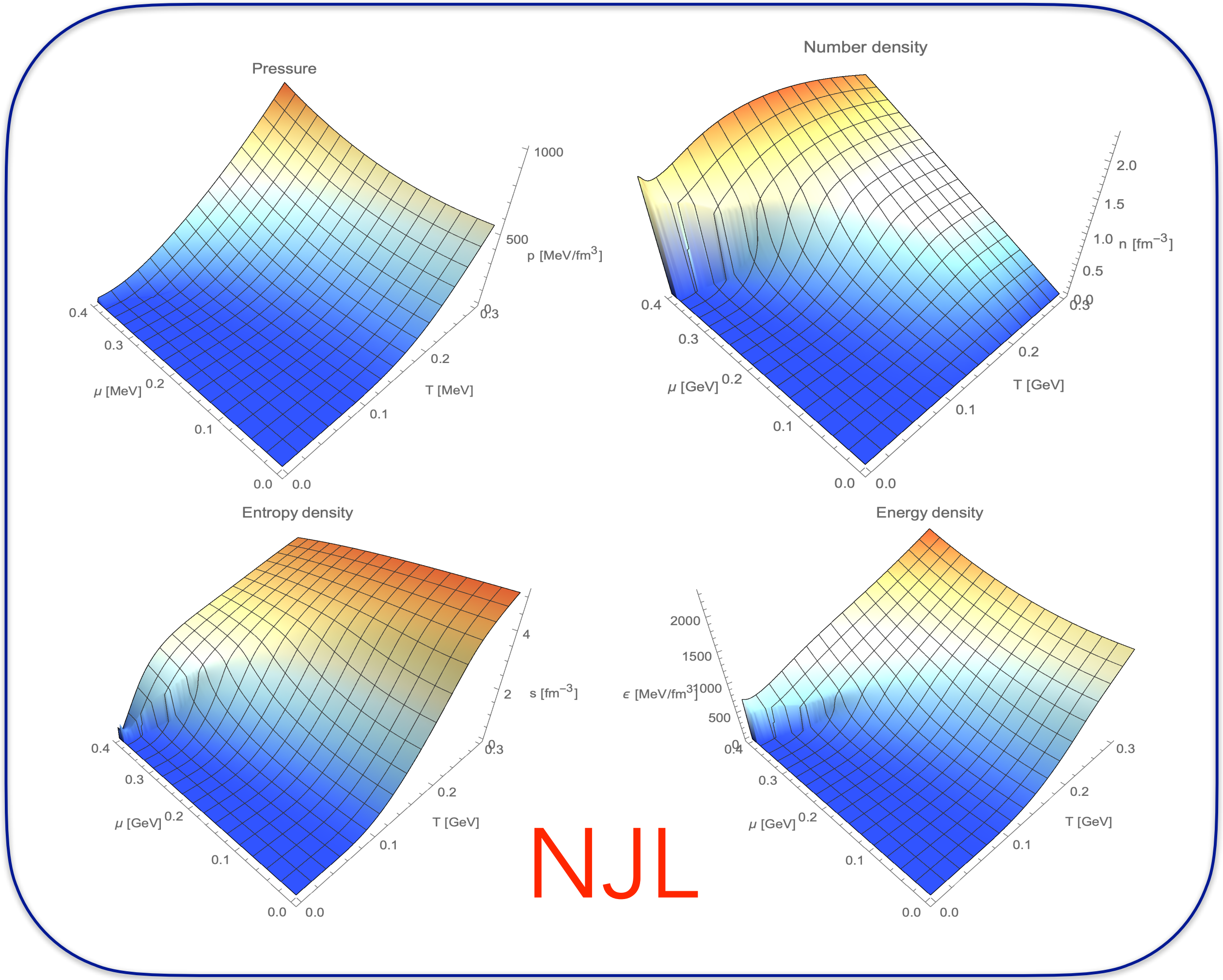
Thermodynamic properties : NJL vs. LIM

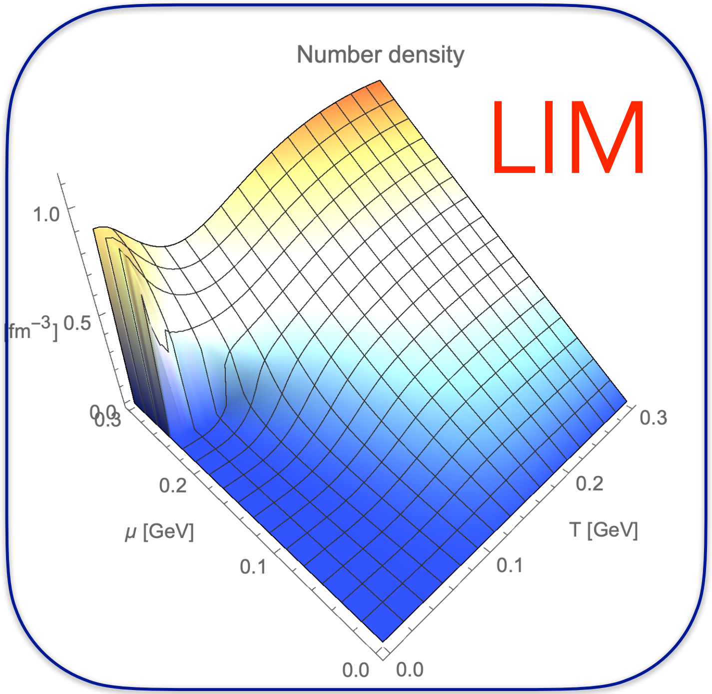




Chiral phase diagram via effective quark mass

Thermodynamic properties: NJL vs. LIM



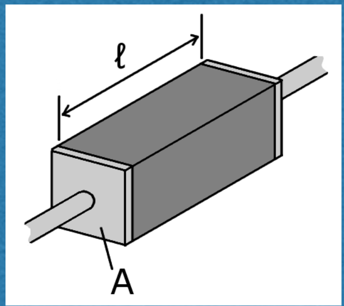


4. Some numerical results

Various transport coefficients

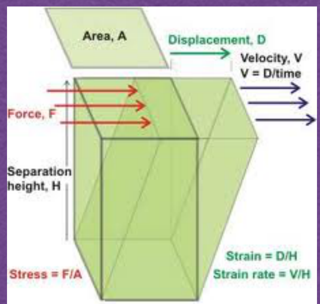
Kubo formula: Current-current correlation

Electric conductivity



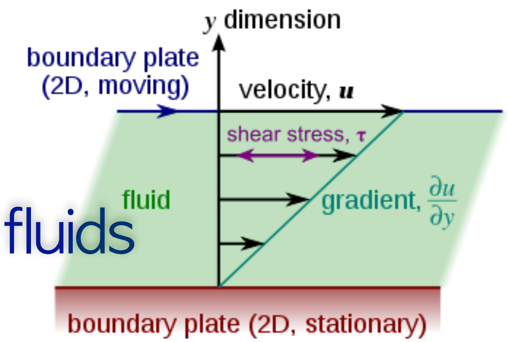
Heat conductivity

Shear viscosity



Bulk viscosity

4. Some numerical results



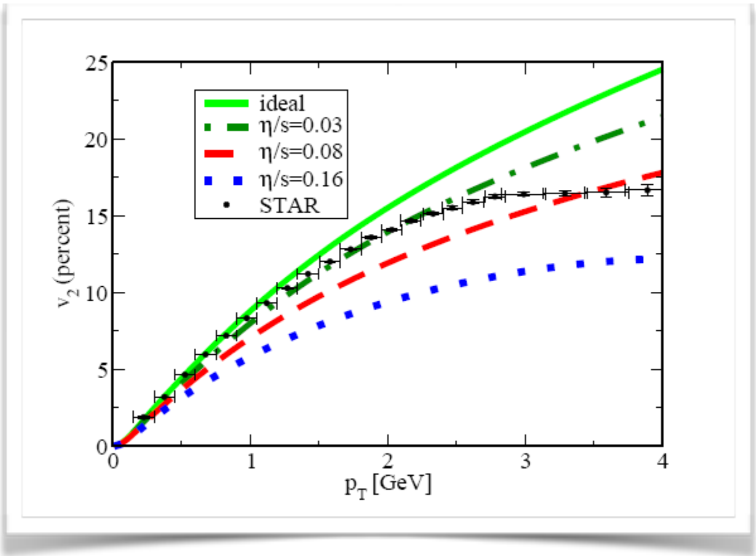
Shear viscosity (η)

η amounts strength of the shear force for fluids

$$\frac{F}{A} = \eta \frac{u}{y}$$

Small shear viscosity means "not sticky"

In viscous hydrodynamics simulations, η of QGP used as a parameter



Small η

Large η

1.Introduction: Lattice QCD

- QCD is a first principle for strong interactions but too difficult in low-E as we have seen
- Ideas for overcoming huddles:

1) We have computing machines
2) Physics is based on CALCULUS
3) Correlations can be expressed by multiple differentiations
4) Reconstruct QCD in discret spacetime

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

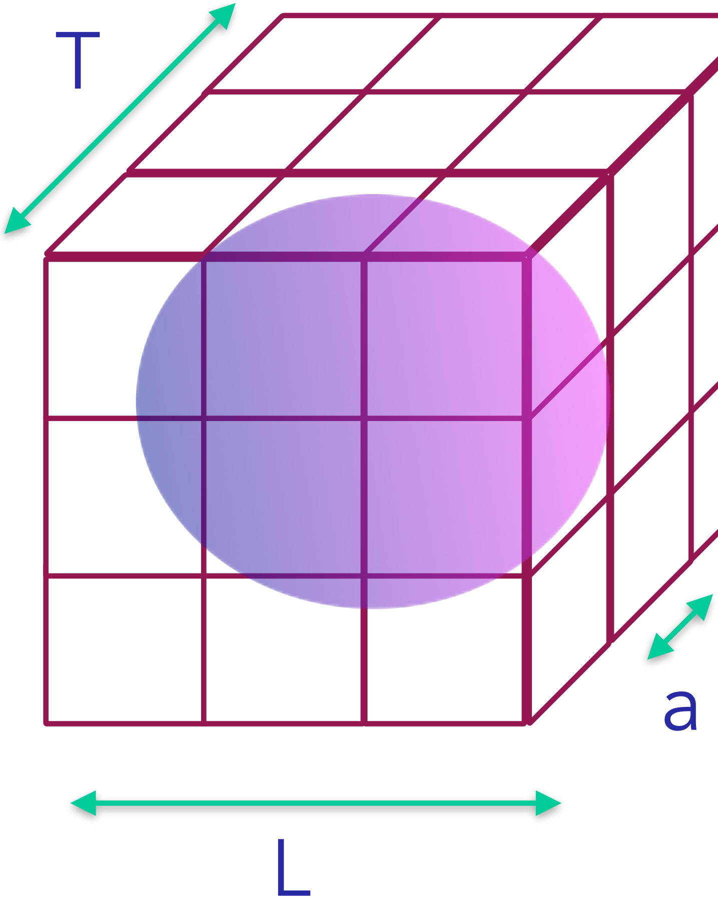
5) Using path integral for correlations and statistical methods: Why????
6) Profit!!

Kenneth G. Wilson (1936 ~ 2013)

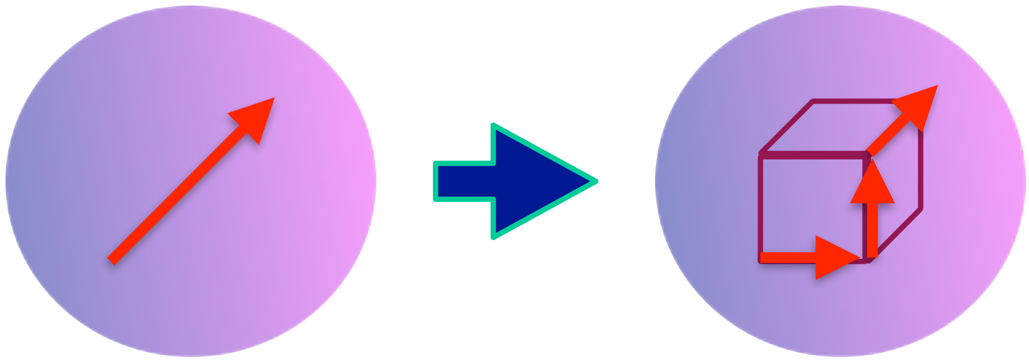
Physical Review D. **25** (10): 2649.

1.Introduction: Lattice QCD

QCD correlation functions are redefined in discretized space-time



Four-dimensional Euclidean space-time with volume L^3T



$\langle 0|O(x)O(y)|0\rangle$

In continuous limit a → 0,
it becomes our world again

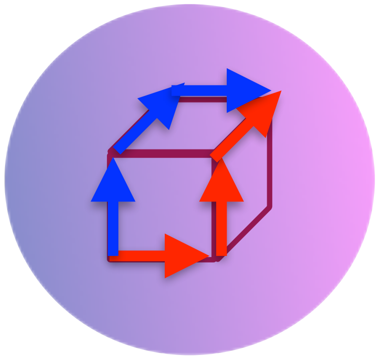
1.Introduction: Lattice QCD

- Unfortunately, we have infinite possible paths as quantum fluctuations: Which route do I need to take?

- We have a powerful method for this: Path integral

- Ok, fine, then how to perform path integral with the discrete spacetime technically?

- Again, we have powerful method:
  Statistical Monte-Carlo simulation



Stanisław Marcin Ulam

1.Introduction: Lattice QCD

- First, we start with the path integral for this purpose for QCD

$$\langle \mathcal{O}(\bar{\psi}, \psi, U) \rangle = \frac{1}{Z} \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi \; \mathcal{O}(\bar{\psi}, \psi, U) \; \mathrm{e}^{-S_G[U] - S_F[\bar{\psi}, \psi, U]}$$

- Using external Grassmann fields to integrate out the fermion fields

$$\langle \mathcal{O}(\bar{\psi}, \psi, U) \rangle = \frac{1}{Z} \int \mathcal{D}U \; (\det D(U)) \; \mathrm{e}^{-S_G[U]} \; \mathcal{O}'(U)$$

Redefined operator $\mathcal{O}'(U) \equiv \mathcal{O}(-\frac{\partial}{\partial \eta}, \frac{\partial}{\partial \bar{\eta}}, U) \; \mathrm{e}^{\bar{\eta} D^{-1}(U) \eta} \Big|_{\eta = \bar{\eta} = 0}$

1.Introduction: Lattice QCD

How to perform MC with this???

1. Generate a uniform random number i
2. Generate a gauge configuration Ui by weighting probability P=det[D(Ui)] exp(-SG[Ui]) to the uniform random number

Importance sampling: P and 1/P are known!!

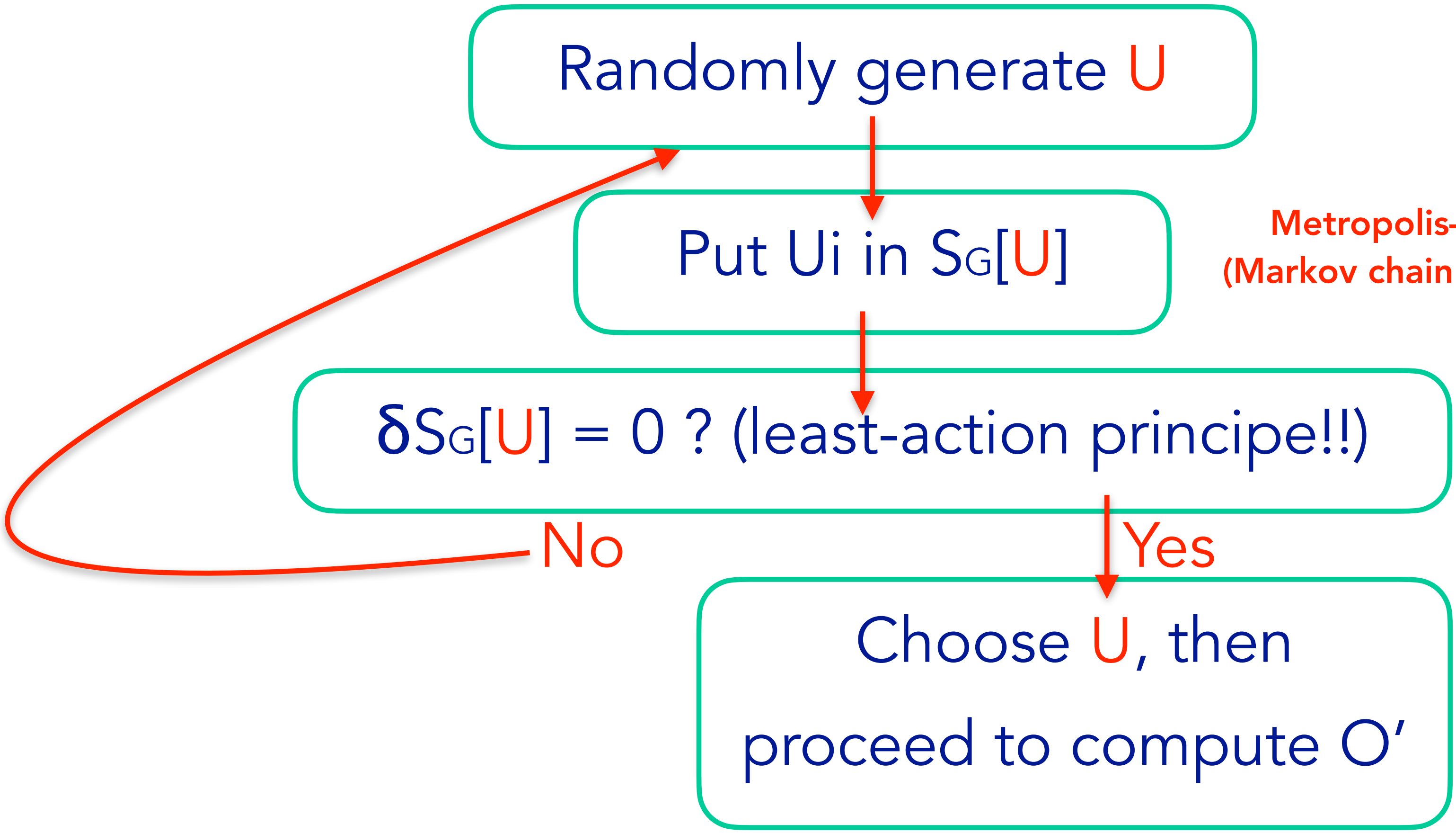
3. Calculate O'(Ui) for the obtained Ui
4. Repeat the process N times

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \mathcal{O}'(U_i) = \frac{1}{Z}\int \mathcal{D}U \; \mathrm{e}^{-S_G[U]} \; \mathcal{O}'(U) = \langle \mathcal{O}(\bar{\psi}, \psi, U)\rangle$$

Generating U_i with P

1.Introduction: Lattice QCD

- Sequential generating U via Markov-Chain MC
- Metropolis-Hastings algorithm: Certain probability of $U_i \rightarrow U_j$



Metropolis–Hastings algorithm
(Markov chain Monte Carlo (MCMC))

Quenched!

1.Introduction: Lattice QCD

- Make things easy! : Quenched approximation

- There are infinite sea (virtual) quarks in Dirac sea: Quark loops

- Decoupling sea quarks by making sea-quark mass infinite

$$\frac{\int \mathcal{D}U \ (\det D(U)) \ \mathrm{e}^{-S_G[U]} \ \mathcal{O}'(U)}{\int \mathcal{D}U \ (\det D(U)) \ \mathrm{e}^{-S_G[U]}} \sim \frac{\int \mathcal{D}U \ (\det \cancel{D}(U)) \ \mathrm{e}^{-S_G[U]} \ \mathcal{O}'(U)}{\int \mathcal{D}U \ (\det \cancel{D}(U)) \ \mathrm{e}^{-S_G[U]}}$$

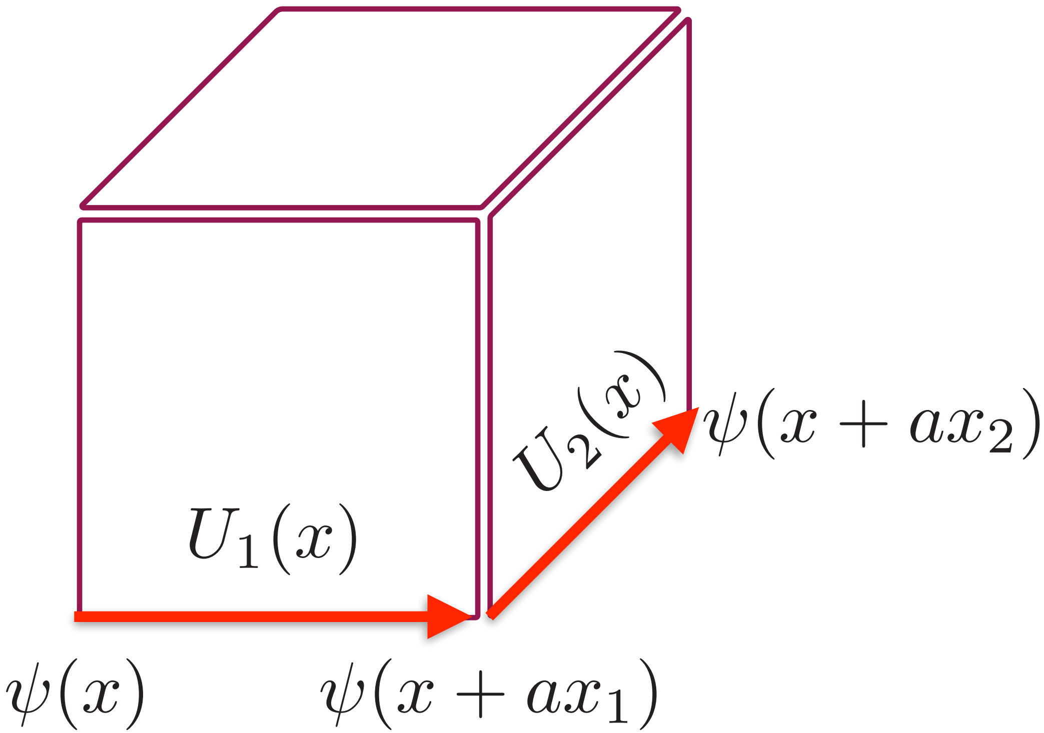
- This treatment is the same with det D(U) =1

- Due to this, "P" becomes local (without derivatives) and simple!!!!

Quenched!

1.Introduction: Lattice QCD

▪How to make S_G in LQCD? : Plaquette action



▪Link variable U which make (anti)quark move to a next site

$$U_\mu(x) = \exp\left[iaA_\mu(x)\right]$$

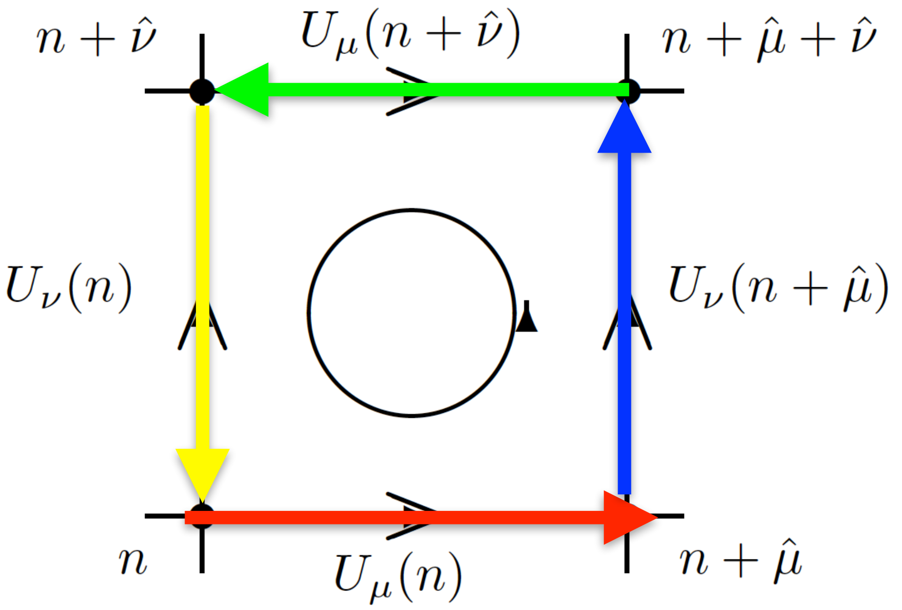
▪U can be understood as a gauge link in SU(Nc)

$$G(x,y) = P\exp\left[i\int A_\mu ds^\mu\right]$$

Quenched!

1.Introduction: Lattice QCD

- What is a gauge-invariant quantity, constructed by U?
- A smallest closed loop L of multiplications of U: Plaquette



I do not prove Plaquette is gauge invariant..

$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+ax_\mu)U^\dagger_\mu(x+ax_\nu)U^\dagger_\nu(x)$$

Quenched!

1.Introduction: Lattice QCD

- Constructing action with Plaquette: Wilson gauge action

$$S_G[U] = \frac{2}{g^2} \sum_x \sum_{\mu<\nu} \mathrm{ReTr}\left[1 - U_{\mu\nu}(x)\right]$$

I do not prove equivalence..

- In continuous limit, it (closely) becomes usual QCD gauge action

- In SU(2), this action can be written as

$$S_P[U] \;=\; \beta \sum_x \sum_{\mu=1}^{3} \left[(4-\mu) - \frac{2}{N_c} b_\mu^0(x) \right] \qquad \sum_{a=0}^{3} \left(b_\mu^a(x)\right)^2 = 1$$

- Here, we have used the SU(2) generator nature (Pauli matrix)

- After tedious calculations, we arrive at the final expression:

$$\left\langle \mathcal{O}(\bar{\psi}, \psi, U) \right\rangle \;=\; \frac{1}{Z} \prod_{x,\,\mu} \mathrm{e}^{-(4-\mu)} \int_{-1}^{1} d\left(\cos\theta\right) \int_{0}^{2\pi} d\phi \int_{-1}^{1} db_\mu^0(x) \, \frac{\sqrt{1-\left(b_\mu^0(x)\right)^2}}{2} \exp\left[\frac{2\beta}{N_c} b_\mu^0(x)\right] \mathcal{O}'(U)$$

Quenched!

1.Introduction: Lattice QCD

■SU(2) Willson (plaquette) action gets simpler

$$\langle \mathcal{O}(\bar{\psi}, \psi, U) \rangle = \frac{1}{Z} \prod_{x,\mu} \mathrm{e}^{-(4-\mu)} \int_{-1}^{1} d(\cos\theta) \int_{0}^{2\pi} d\phi \int_{\mathrm{e}^{-2\beta/N_c}}^{\mathrm{e}^{2\beta/N_c}} dY \quad \frac{N_c}{4\beta} \sqrt{1 - \left(\frac{N_c}{2\beta} \log Y \right)^2} \; \mathcal{O}'(U)$$

$$Y = \exp\left[\frac{2\beta}{N_c} b_\mu^0(x) \right] \iff b_\mu^0(x) = \frac{N_c}{2\beta} \log Y$$

Pseudo-Heat-bath method (importance sampling)

1. Random generation of Y (~b) and 0≦ξ≦1
2. Computing P = √~ then compare it with ξ
3. If P ≧ ξ, take Y (~b), and vice versa going to 1 again
4. Computing O'(U) with obtained Y
5. Generating angles randomly then perform integration!!

2. Application Lattice QCD

Although we have a big jump….

- LQCD in finite quark chemical potential: What's wrong with this?
  - We comput $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \mathcal{O}'(U_i)$ with P = $(\det D(U))\ e^{-S_G[U]}$

Note that det[D] = DD⁺

- The quark Dirac operator with chemical potential reads

$$D(\mu_q) = \not{D} + m + \mu_q\gamma_0$$

$$D^\dagger(\mu_q) = -\not{D} + m + \mu_q^*\gamma_0 = \gamma_5 D^\dagger(-\mu_q^*)\gamma_5$$

$$\{\det[D(\mu_q)]\}^* = \det[D^\dagger(\mu_q)] = \det[\gamma_5 D(-\mu_q^*)\gamma_5] = \det[D(-\mu_q^*)]$$

- If μ is real, det[D] is not real (complex), and VICE VERSA

det[D] must be real, since it is probability P!!!

2. Application Lattice QCD

- In addition, if it is a complex, then we have

$$\int dU O'(U)(R+iI)e^{-S_G} \sim \int dU O'(U)e^{-S_G+i\phi}$$

- It's oscillation to cancel out the integral: Sign problem
- Notorious problem in strongly interacting fermion systems even in condensed matter, QFT, and nuclear physics as well.

- How to solve the sign problem???
- So far, there have been no cures (NP-hard problem)
- Many indirect and approximated methods developed

2. Application Lattice QCD

- Canonical approach developed!!

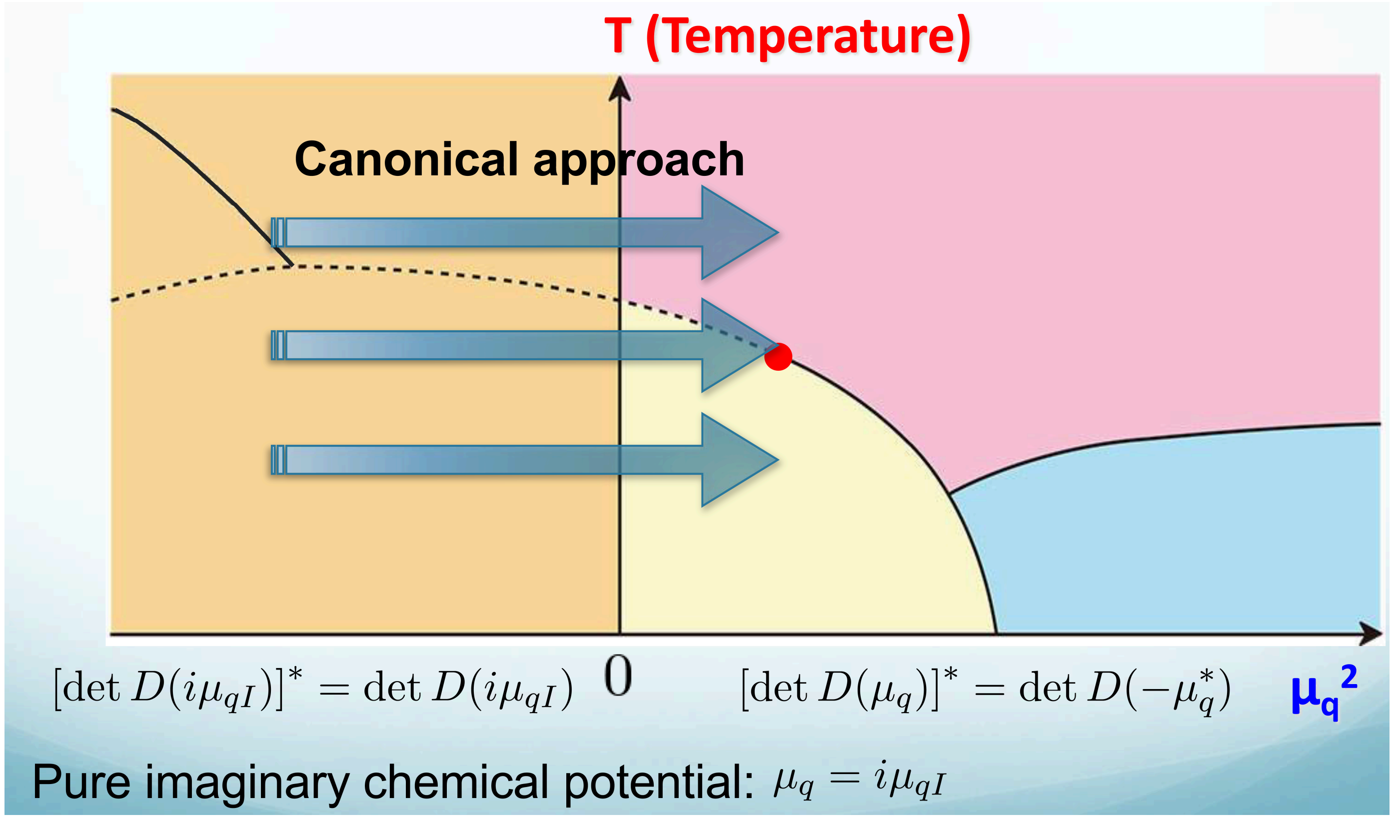


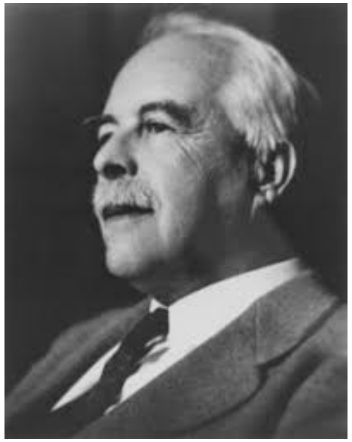
Figure by Dr. Wakayama

2. Application Lattice QCD

- Fugacity expansion of grand canonical partition function

$$Z_{GC}[\mu_q, T, V] = \sum_n Z_C[n, T, V]\xi^n, \quad \xi = e^{\mu_q/T}$$

Fugacity


Gilbert Newton Lewis

- Obtain canonical function partition function by Fourier transform

$$Z_C[n, T, V] = \int_0^{2\pi} \frac{\mu_{qI}/T}{2\pi} e^{-n\mu_{qI}/T} Z_{GC}[\mu_{qI}, T, V]$$

- For imaginary chemical potential, there is no SIGN problem

One can do MCMC or Metropolis-Hastings MC

Then, we obtain Z_{GC} on LQCD

2. Application Lattice QCD

Canonical approach developed

Lattice QCD

$$n_q(\mu_q = i\mu_{qI}, T, V)$$

Number density formulation
V. Bornyakov et al., PRD95(2017)

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\mathrm{GC}}$$

Like Hohenberg-Kohn theorem??

$$Z_{\mathrm{GC}}(\mu_q = i\mu_{qI}, T, V)$$

Fourier transform

$$Z(n, T, V)$$

$$Z_{\mathrm{GC}}(\mu_q, T, V) = \sum_{n=-\infty}^{\infty} Z(n, T, V)\xi^n$$

$$\xi = e^{\mu_q/T}$$

If we get Z_n for all n, we can search at ANY density!

Slide by Dr. Wakayama

2. Application Lattice QCD

Canonical approach

Lattice QCD

$$n_q(\mu_q = i\mu_{qI}, T, V)$$

Number density formulation
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$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC}$$

$$Z_{\rm GC}(\mu_q = i\mu_{qI}, T, V)$$

Fourier transform

$$Z(n, T, V)$$

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n \qquad \xi = e^{\mu_q/T}$$

In numerical calculations, n is finite.

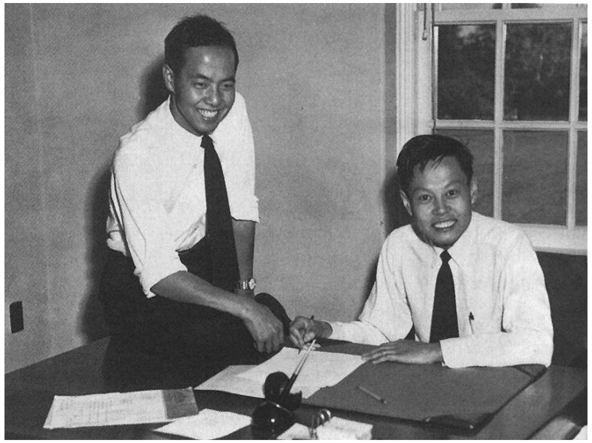
Slide by Dr. Wakayama

2. Application Lattice QCD

- Application of canonical method: **Lee-Yang zeros**

- Zeros of Z_{GC} so-called Lee-Yang Zeros (LYZ) contain a valuable information on the phase transitions of a system.

T.D. Lee & C.N. Yang, Phys. Rev. 87, 404&410 (1952)



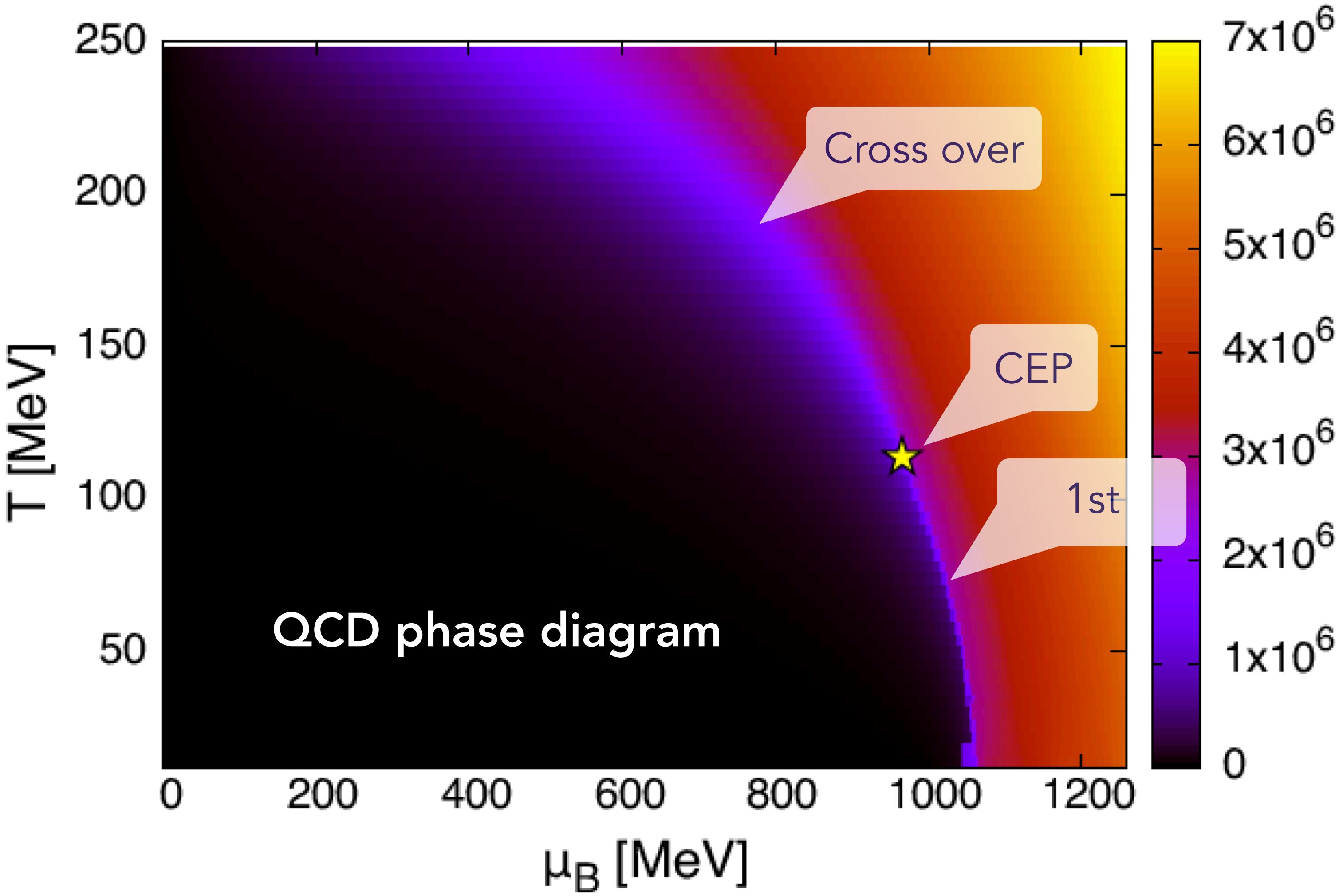
$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-N_{\rm max}}^{N_{\rm max}} Z_{\rm c}(n, T, V)\xi^n = 0$$

Physically, at LYZ, critical-end point (CEP) appears!!

2. Application Lattice QCD

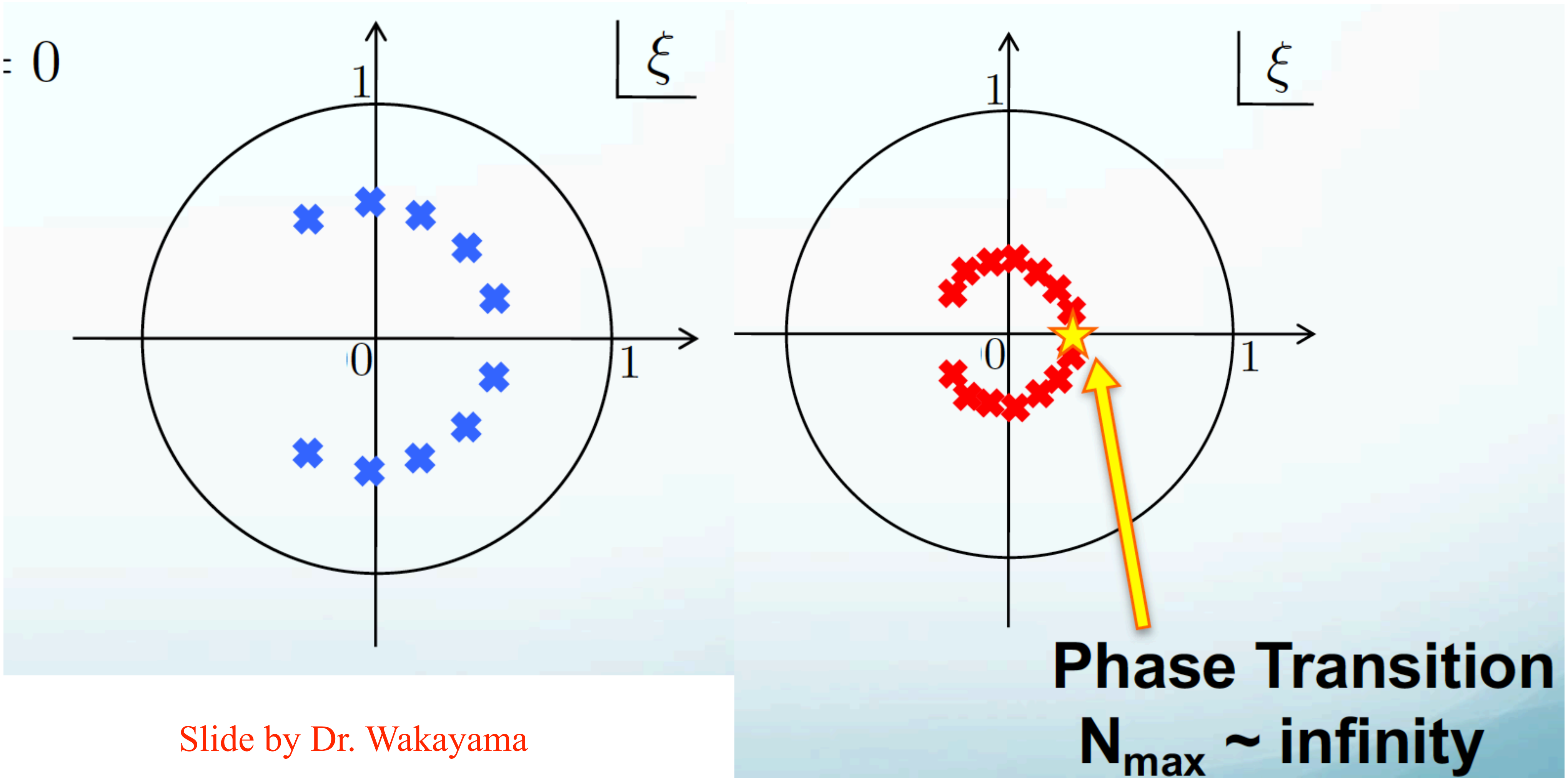
Application of canonical method: **Lee-Yang zeros**

What is critical-end point (CEP)??



2. Application Lattice QCD

- Application of canonical method: **Lee-Yang zeros**
  - There are $2N_{max}$ LYZs in complex fugacity plane



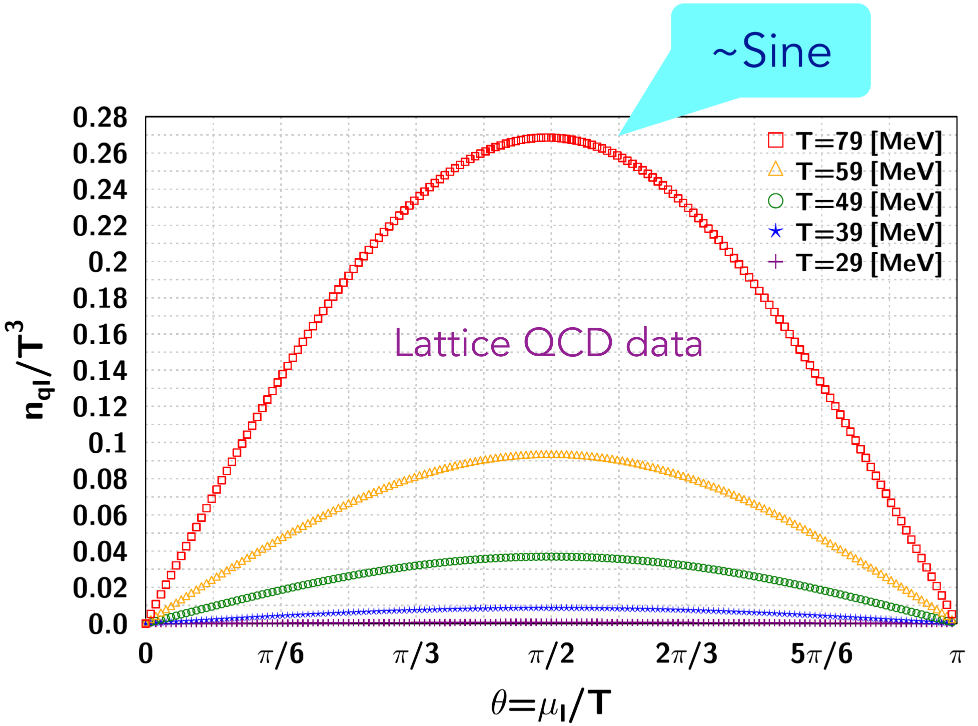
Slide by Dr. Wakayama

2. Application Lattice QCD

- Application of canonical method: **Lee-Yang zeros**
- First, we parameterize number density with sine function for more reliable numerical treatment in lattice QCD

$$\frac{n_{qI}}{T^3}(\theta) = \sum_{k=1}^{N_{\sin}} f_k \sin(k\theta)$$

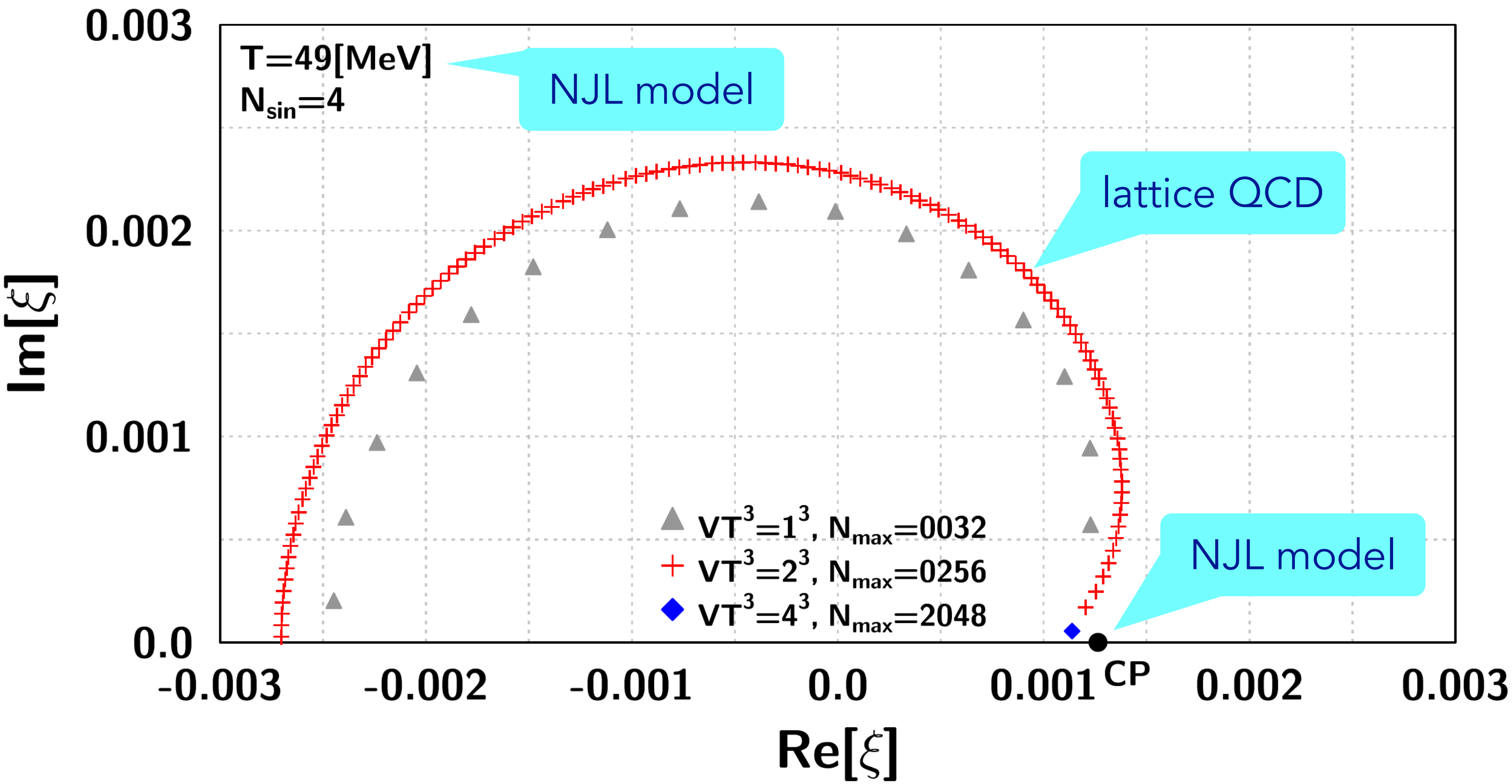
$$Z_{\rm GC}(i\mu_I, T, V) = C\exp\left\{-V\int_0^{\theta} d\theta'\, n_{qI}(\theta')\right\} = C\exp\left\{VT^3\sum_{k=1}^{N_{\sin}}\frac{f_k}{k}\cos(k\theta)\right\},$$



Wakayama and Hosaka, PRD (2019)

2. Application Lattice QCD

- Application of canonical method: **Lee-Yang zeros**
  - We observe LYZs cross the Im[ξ]=0 line: CEP



Wakayama and Hosaka, PRD (2019)

2. Application Lattice QCD

- Application of canonical method: **QCD phase structure**
- This method is not full lattice QCD but mimics it closely
- Then, can we describe QCD phase diagram???: Yes!!!
- Before doing lattice QCD with canonical method,
- we test it in effective models: NJL and PNJL

Thermodynamic potential of PNJL

$$\omega = \frac{1}{2G}(M-m_q)^2 - 2N_cN_f\int\frac{d^3p}{(2\pi)^3}E_p \; -2N_fT\int\frac{d^3p}{(2\pi)^3}\left\{\mathrm{Tr}_c\ln\left[1+Le^{-\frac{E_p-\mu}{T}}\right]\right.$$

$$\left.+\mathrm{Tr}_c\ln\left[1+L^\dagger e^{-\frac{E_p+\mu}{T}}\right]\right\} + T^4\left[-\frac{b_2(T)}{2}\ell\bar{\ell} - \frac{b_3}{6}\left(\ell^3+\bar{\ell}^3\right) + \frac{b_4}{4}\left(\ell\bar{\ell}\right)^2\right]$$

Quark

Mass gap

Quark-Gluon

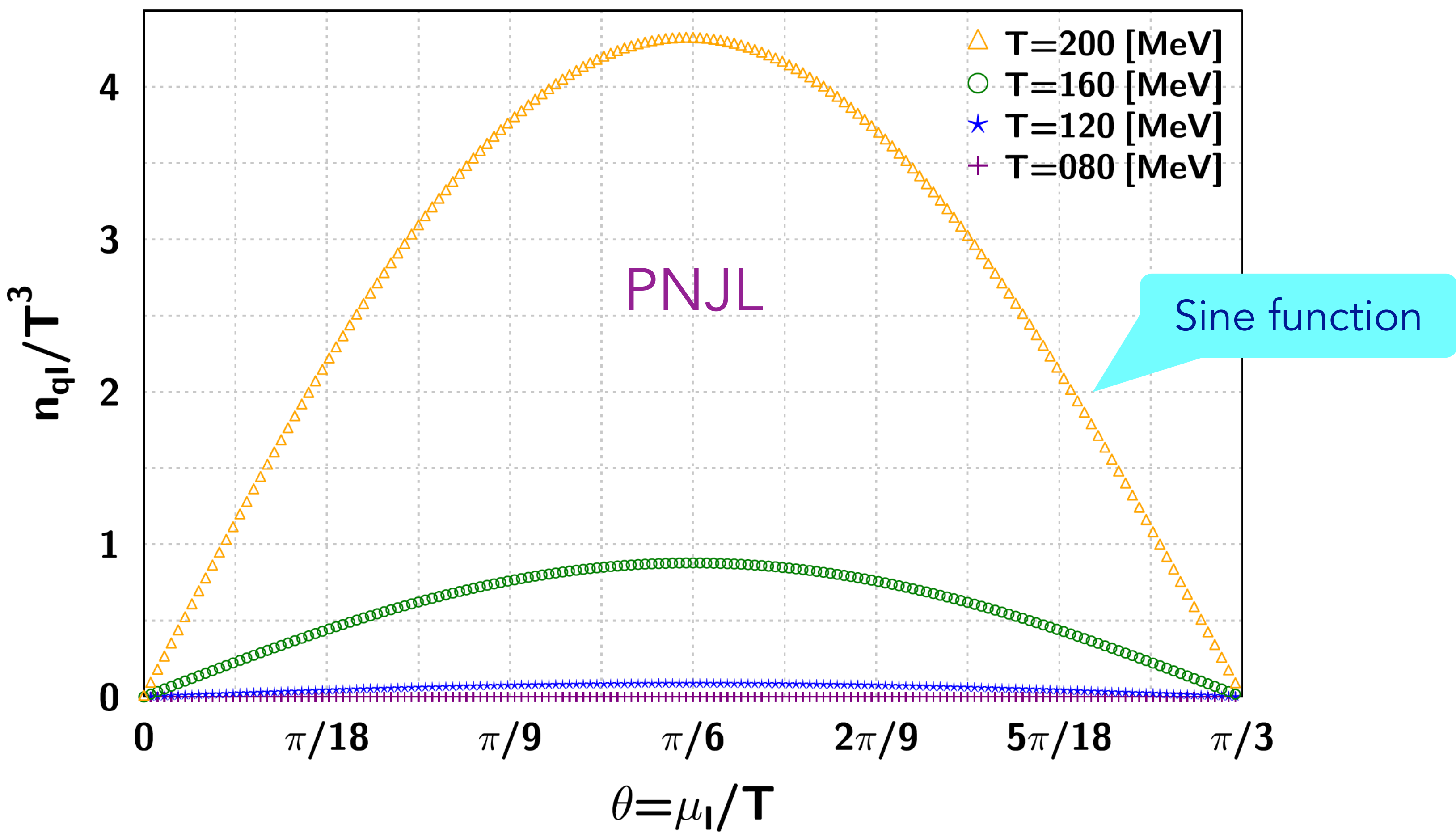
Quark-Gluon

Gluon ~ Z(Nc)

Wakayama, Nam, and Hosaka, PRD (2020)

2. Application Lattice QCD

Application of canonical method: **QCD phase structure**

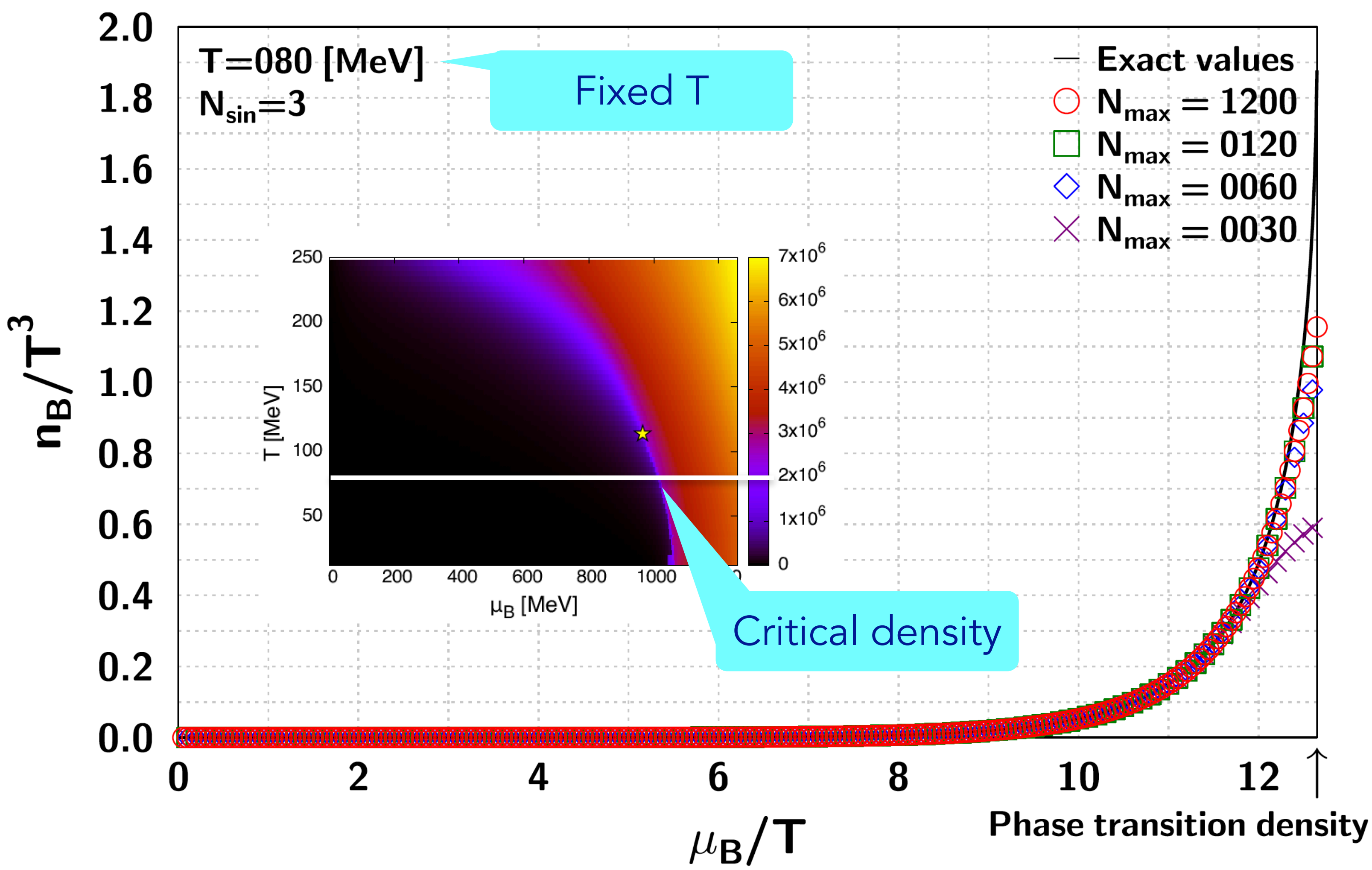


Wakayama, Nam, and Hosaka, PRD (2020)

2. Application Lattice QCD

Wakayama, Nam, and Hosaka, PRD (2020)

Application of canonical method: **QCD phase structure**



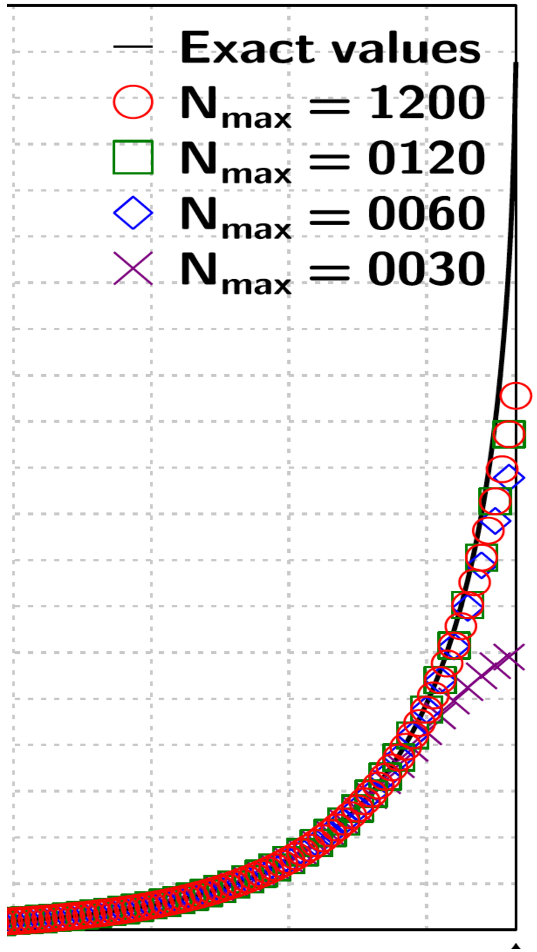
2. Application Lattice QCD

Wakayama, Nam, and Hosaka, PRD (2020)

- Application of canonical method: **QCD phase structure**

- As N_{max} increases, results from canonical method reaches to exact value
- Nonetheless, canonical method does not coincide with exact one: limitation of the method…
- Then, how do we quantify phase transition in this method?: Taking tolerance

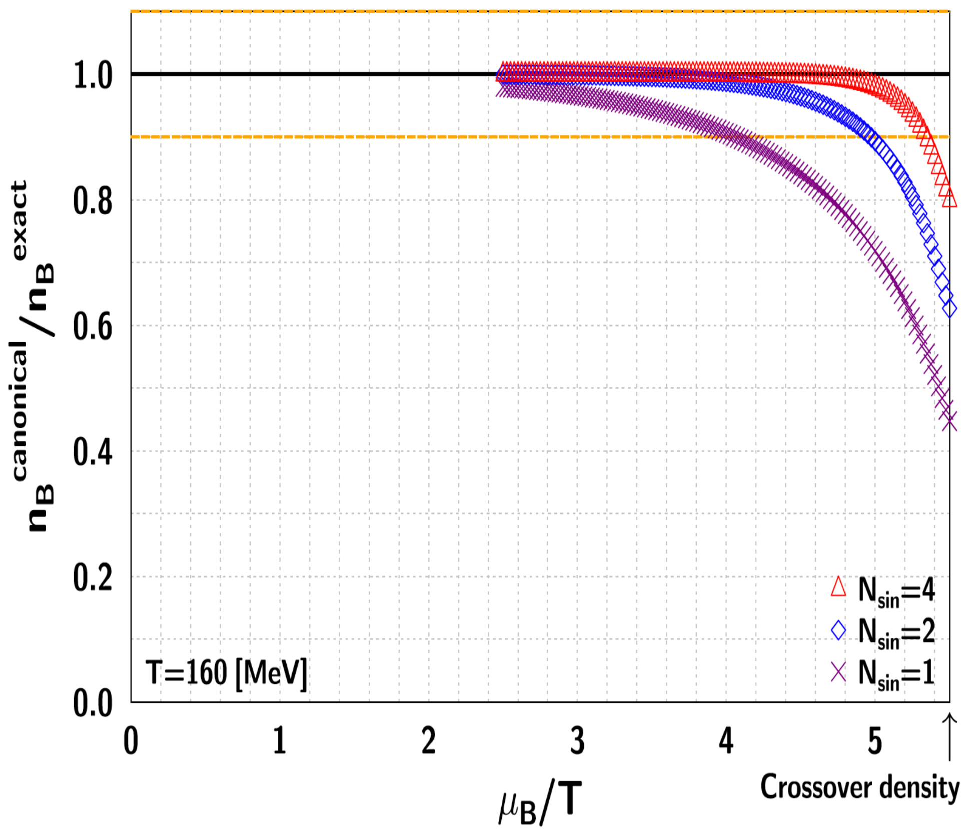
$$\frac{n_B^{\mathrm{PNJL}}}{n_B^{\mathrm{Canonical}}} < 10\%$$

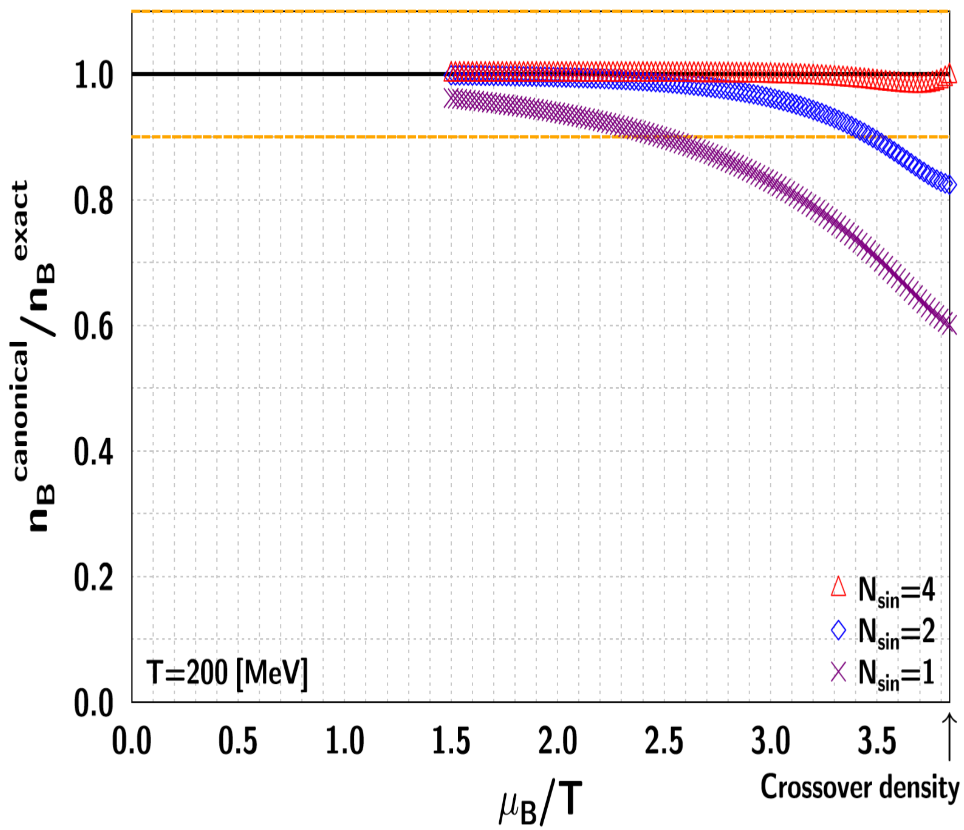


2. Application Lattice QCD

Wakayama, Nam, and Hosaka, PRD (2020)

■Application of canonical method: **QCD phase structure**

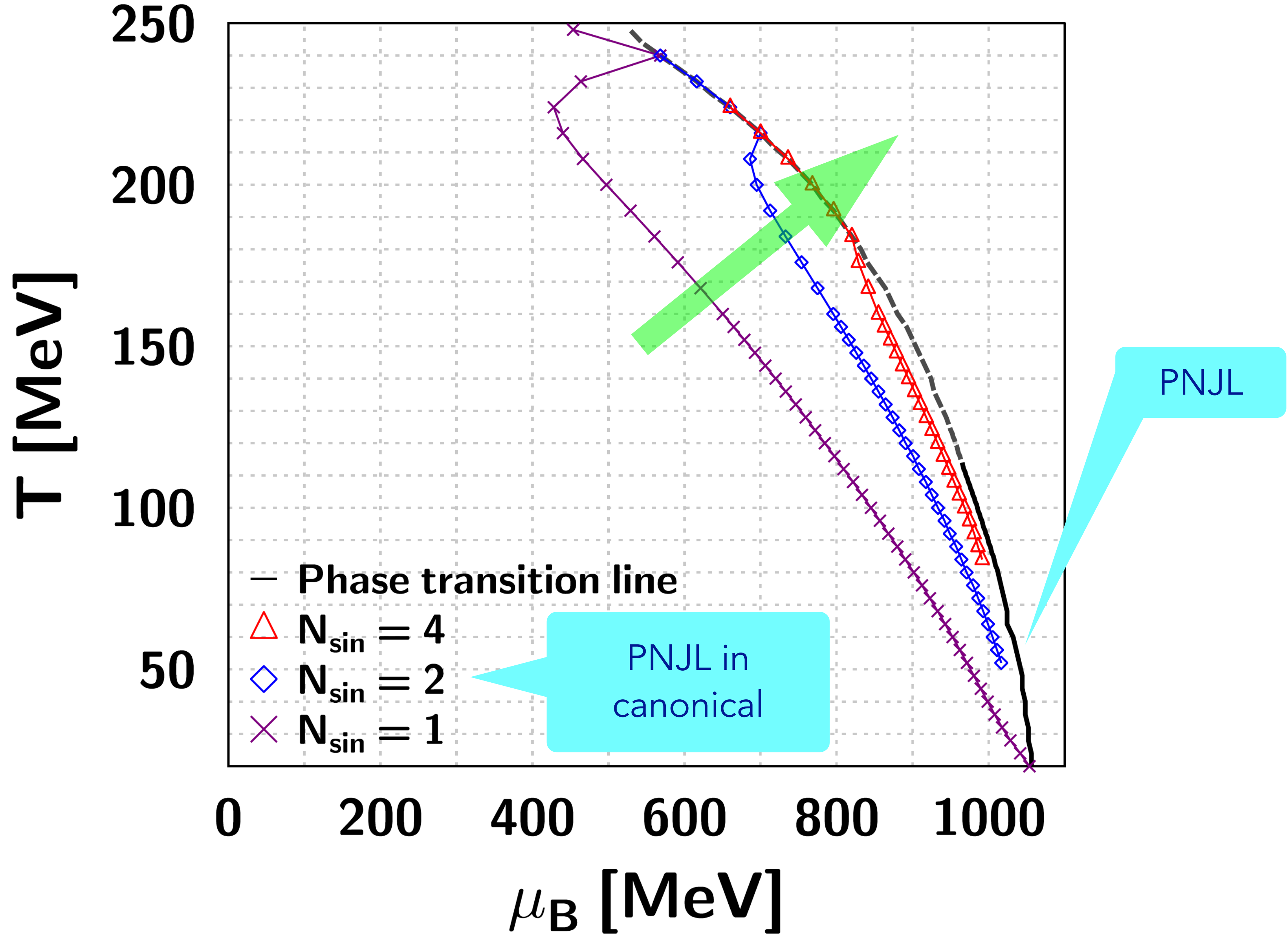




$$\frac{n_B^{\rm PNJL}}{n_B^{\rm Canonical}} < 10\%$$

2. Application Lattice QCD

Wakayama, Nam, and Hosaka, PRD (2020)



Thank you for your attention!!