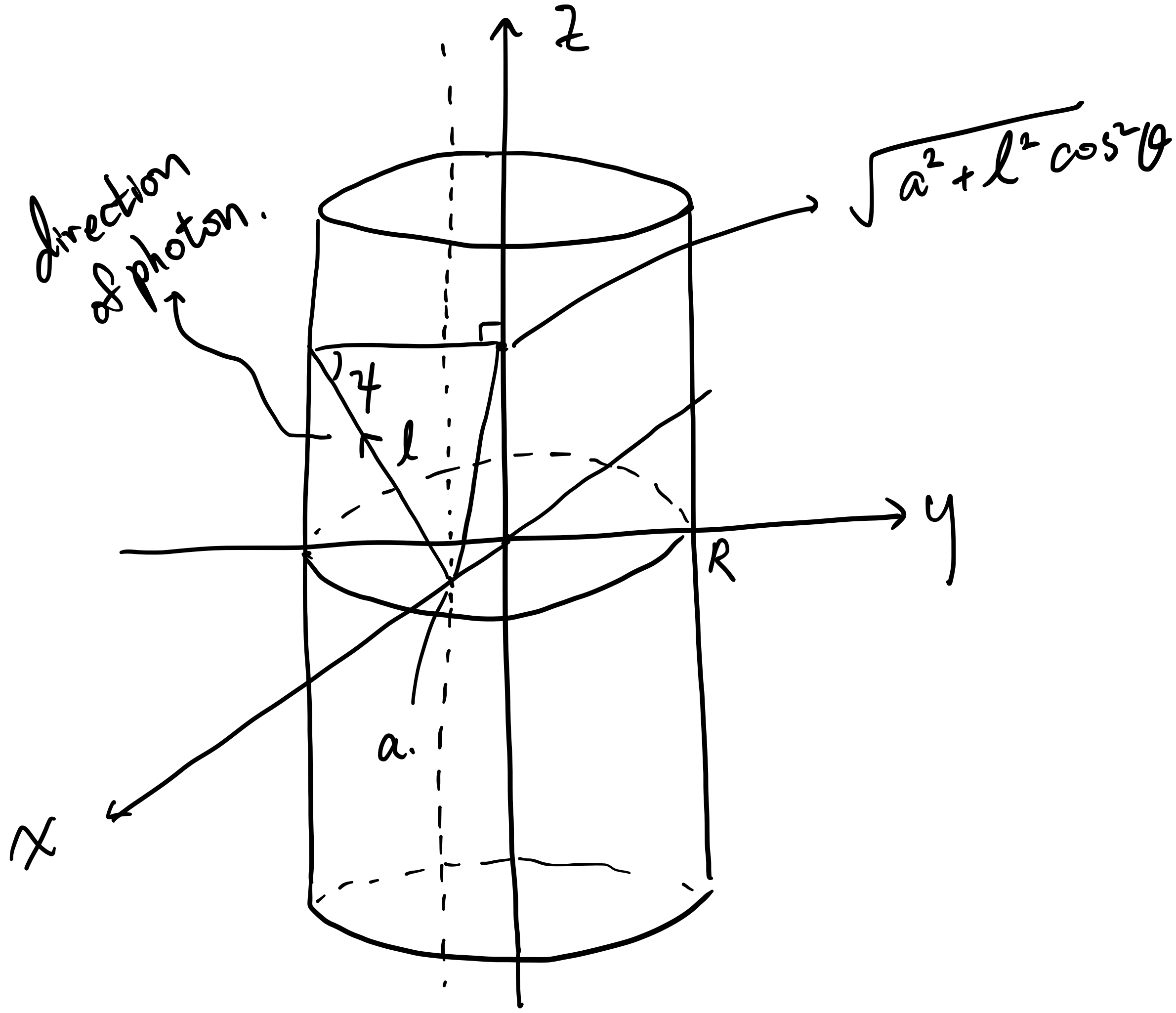
# Probability

for total reflection

in fiber



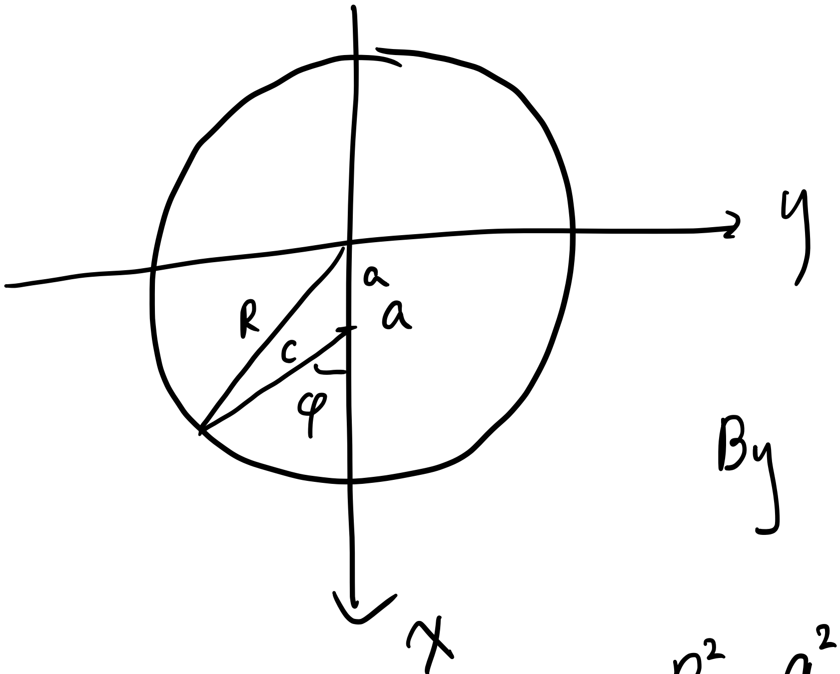

$\Rightarrow$ For total reflection,

$$\psi > 69.57^\circ = \psi_c$$

direction of photon이 z축과 이루는 각 : $\theta$

" " 의 xy평면 정사영이

x축과 이루는 각 : $\varphi$




By 2nd cosine law

$$R^2 = a^2 + c^2 - 2ac\underbrace{\cos(\pi-\varphi)}_{= -\cos\varphi}$$

$$\therefore\ c^2 + (2a\cos\varphi)c + a^2 - R^2 = 0$$

$$\therefore\ c = -a\cos\varphi \pm \sqrt{a^2\cos^2\varphi - a^2 + R^2}$$

$$= -a\cos\varphi \pm \sqrt{R^2 - a^2\sin^2\varphi}$$

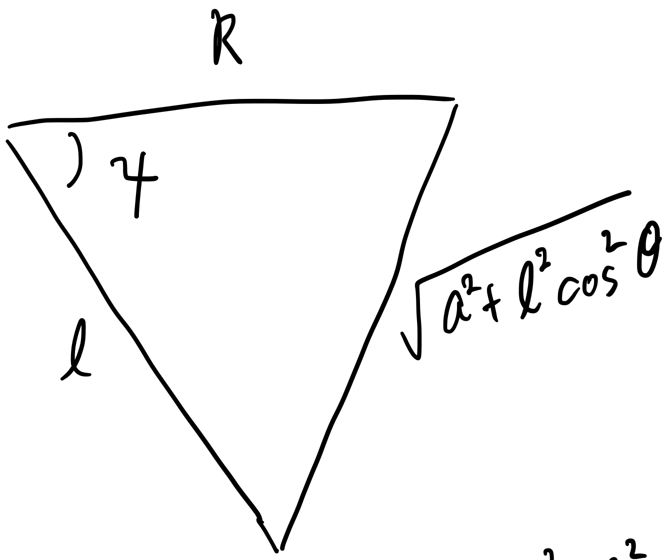
$$= c(\varphi).$$

because $c(\varphi) > 0$

$$\therefore\ c(\varphi) = -a\cos\varphi + \sqrt{R^2 - a^2\sin^2\varphi}$$

And $l\sin\theta = C(\varphi)$ 이므로

$$l = \frac{C(\varphi)}{\sin\theta}$$




$$\therefore \cos\psi = \frac{R^2+l^2-(a^2+l^2\cos^2\theta)}{2Rl}$$

$$= \frac{R^2+l^2\sin^2\theta - a^2}{2Rl} < \cos\psi_c$$

and $l = \dfrac{C(\varphi)}{\sin\theta}$ 이므로

$$\frac{(R^2+C(\varphi)^2-a^2)\sin\theta}{2RC(\varphi)} < \cos\psi_c$$

$$\therefore \theta < \sin^{-1}\left\{ \frac{\frac{R}{a}\cos\psi_c\left(\sqrt{\left(\frac{R}{a}\right)^2-\sin^2\varphi}-\cos\varphi\right)}{\left(\frac{R}{a}\right)^2-\sin^2\varphi-\cos\varphi\sqrt{\left(\frac{R}{a}\right)^2-\sin^2\varphi}} \right\}$$

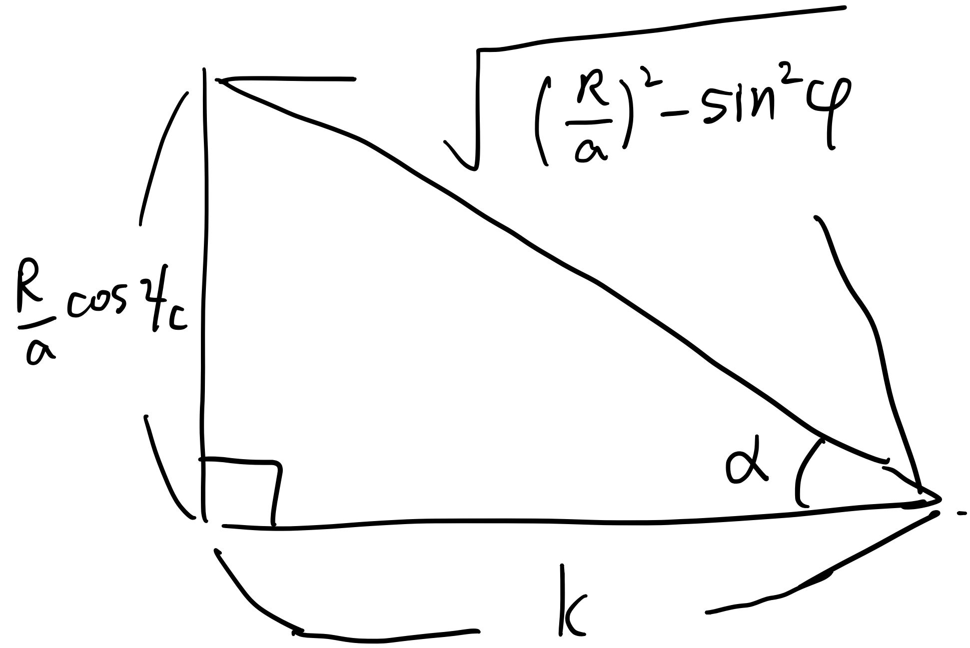
$$= \sin^{-1}\left\{ \frac{\frac{R}{a}\cos\psi_c}{\sqrt{\left(\frac{R}{a}\right)^2-\sin^2\varphi}} \right\} = f(\varphi)$$

$\therefore$ At point $(a, 0, 0)$

probability for total reflection

$$= \frac{\int_0^{2\pi}\int_0^{f(\varphi)} \sin\theta \, d\theta \, d\varphi}{\iint d\Omega}$$

$$= \frac{1}{4\pi}\int_0^{2\pi} 1 - \cos\left[\sin^{-1}\left(\frac{\frac{R}{a}\cos\psi_c}{\sqrt{\left(\frac{R}{a}\right)^2-\sin^2\varphi}}\right)\right] d\varphi$$




$$k = \sqrt{\left(\frac{R}{a}\right)^2 - \sin^2\varphi - \left(\frac{R}{a}\right)^2 \cos^2\psi_c}$$

$$\therefore \quad \cos\alpha = \sqrt{\frac{\left(\frac{R}{a}\right)^2 - \sin^2\varphi - \left(\frac{R}{a}\right)^2\cos^2\psi_c}{\left(\frac{R}{a}\right)^2 - \sin^2\varphi}}$$

$$\therefore \quad \frac{1}{4\pi}\int_0^{2\pi} 1 - \sqrt{\frac{\left(\frac{R}{a}\right)^2 \sin^2\psi_c - \sin^2\varphi}{\left(\frac{R}{a}\right)^2 - \sin^2\varphi}} \, d\varphi$$

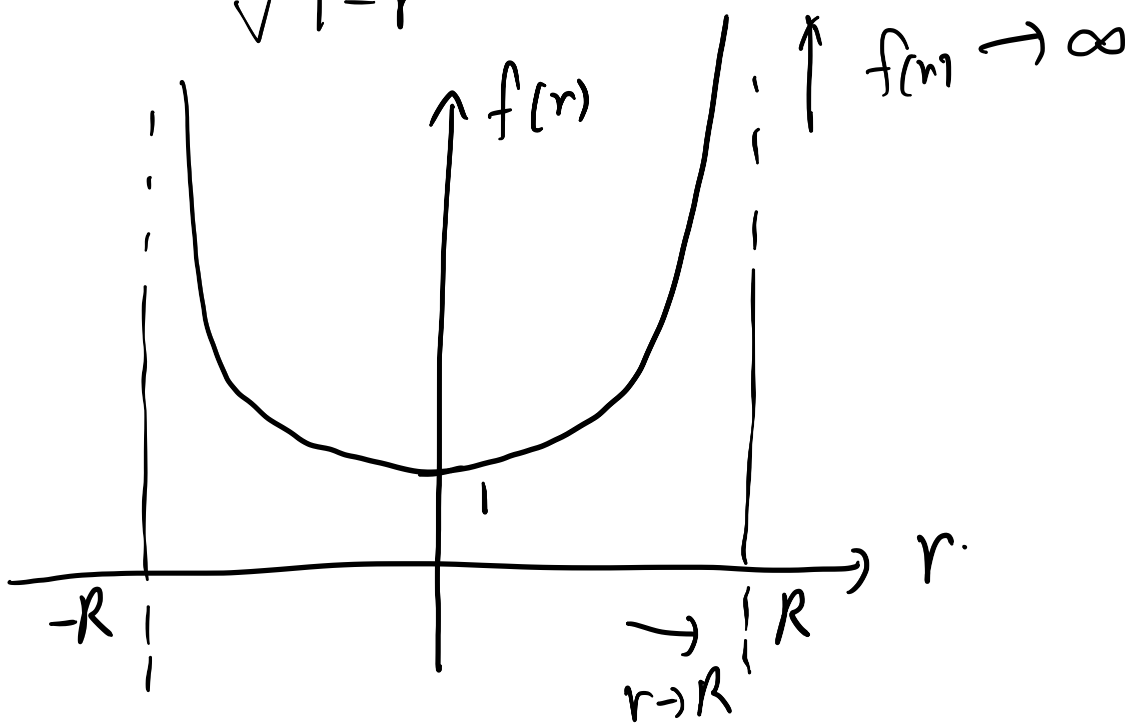
$$\int \frac{a-\sin^2 x}{b-\sin^2 x}\,dx = \frac{(a-b)\tan^{-1}\left(\sqrt{1-\frac{1}{b}}\tan x\right)}{\sqrt{b-1}\,\sqrt{b}} + x$$

$$\therefore \frac{1}{4\pi}\,\frac{\frac{R}{a}\cos^2\psi_c}{\sqrt{\left(\frac{R}{a}\right)^2-1}}\tan^{-1}\left\{\sqrt{1-\left(\frac{a}{R}\right)^2}\tan\varphi\right\}\Bigg|_0^{2\pi}$$

$$= \frac{\frac{R}{a}\cos^2\psi_c}{2\sqrt{\left(\frac{R}{a}\right)^2-1}}$$

$$= \frac{\cos^2\psi_c}{2}\,\frac{1}{\sqrt{1-\left(\frac{a}{R}\right)^2}}$$

$$f(r) = \frac{1}{\sqrt{1-r^2}}$$




$\therefore$ Total reflection occurs in the boundary of fiber.

$$\int \sqrt{\frac{a-\sin^2 x}{b-\sin^2 x}}\, dx$$

$$= \left[ \frac{(a-1)\, F\left\{ i \sinh^{-1}\left( \sqrt{\frac{a-1}{a}} \tan x \,\middle|\, \frac{a(b-1)}{(a-1)b} \right) \right\}}{\sqrt{\frac{a-1}{a}}\,(2a+\cos 2x-1)} \right.$$

$$\left. + \frac{\Pi\left\{ \frac{a}{a-1} ; i \sinh\left( \sqrt{\frac{a-1}{a}} \tan x \right) \,\middle|\, \frac{a(b-1)}{(a-1)b} \right\}}{\sqrt{\frac{a-1}{a}}\,(2a+\cos 2x-1)} \right]$$

$$\times \left\{ -2i \cos^2 x \sqrt{\frac{(a-1)\tan^2 x}{a}+1} \sqrt{\frac{(b-1)\tan^2 x}{b}+1} \sqrt{\frac{a-\sin^2 x}{b-\sin^2 x}} \right\}$$

$\Rightarrow$ ?! T^T

$$\sqrt{\frac{\sin^2\psi_c - \left(\frac{a}{R}\right)^2 \sin^2\varphi}{1-\left(\frac{a}{R}\right)^2\sin^2\varphi}} = \left(1 - \frac{\sin^2\psi_c - 1}{\left(\frac{a}{R}\right)^2\sin^2\varphi - 1}\right)^{\frac{1}{2}}$$

$$= 1 - \frac{1}{2}\,\frac{\sin^2\psi_c - 1}{\left(\frac{a}{R}\right)^2\sin^2\varphi - 1} + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left\{-\frac{\sin^2\psi_c - 1}{\left(\frac{a}{R}\right)^2\sin^2\varphi - 1}\right\}^2$$

$$+ \cdots$$

$$\sin^2\psi_c - 1 \approx 0.07$$

$$(\text{when } \psi_c = 69.57^\circ)$$

$\therefore$ 2nd order 까지 계산

$$\Rightarrow \frac{1}{4\pi}\int_0^{2\pi} \frac{1}{2}\,\frac{\sin^2\psi_c - 1}{\left(\frac{a}{R}\right)^2\sin^2\varphi - 1}\, d\varphi$$

$$= \frac{\cos^2\psi_c}{8\pi}\int_0^{2\pi} \frac{1}{1-\left(\frac{a}{R}\right)^2\sin^2\varphi}\, d\varphi$$

$$\int \frac{1}{a-\sin^2 x}\,dx = \frac{\tan^{-1}\left(\sqrt{1-\frac{1}{a}}\tan x\right)}{\sqrt{a}\,\sqrt{a-1}}$$

$$\Rightarrow \frac{\cos^2 \psi_c}{8\pi}\,\frac{R}{\sqrt{R^2-a^2}}\tan^{-1}\left\{\frac{\sqrt{R^2-a^2}}{R}\tan\varphi\right\}\Bigg|_0^{2\pi}$$

$$= \frac{\cos^2 \psi_c}{8\pi}\,\frac{R}{\sqrt{R^2-a^2}}\,4\tan^{-1}\left\{\frac{\sqrt{R^2-a^2}}{R}\tan\varphi\right\}\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\cos^2 \psi_c}{2\pi}\,\frac{R}{\sqrt{R^2-a^2}}\tan^{-1}\left\{\frac{\sqrt{R^2-a^2}}{R}\tan\varphi\right\}\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\cos^2 \psi_c}{4}\,\frac{R}{\sqrt{R^2-a^2}} = g(a).$$

$$\therefore \quad \frac{\int_0^{2\pi} d\theta \int_0^R a\,da\, g(a)}{\iint a\,da\,d\theta'}$$

$$= \frac{2\pi \cdot \int_0^R \frac{\cos^2 4_c}{4} \frac{R}{\sqrt{R^2-a^2}}\, a\,da}{\pi R^2}$$

$$= \frac{\cos^2 4_c}{2R} \int_0^R \frac{a}{\sqrt{R^2-a^2}}\, da$$

$R^2 - a^2 = t$

$-2a\,da = dt$

$$= \frac{\cos^2 4_c}{2R} \int \frac{1}{\sqrt{t}} \left(-\frac{dt}{2}\right)$$

$$= -\frac{\cos^2 4_c}{4R} \int t^{-\frac{1}{2}}\, dt = -\frac{\cos^2 4_c}{2R} \sqrt{R^2-a^2}\Big|_0^R$$

$$= \frac{\cos^2 4_c}{2}$$

$\therefore$ Probability $= \dfrac{\cos^2 \psi_c}{2}$

And $\psi_c = 69.57°$

$\Rightarrow \quad P = 0.0608$