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Evolution equations for the double parton distributions

Initial conditions and transverse momentum dependence

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Abstract. In the first part of this contribution we discuss the problem of the initial conditions for the evolution of double parton distribution functions (PDFs). We show that one can construct a framework based on the expansion in terms of the Dirichlet functions in which both single and double PDFs satisfy momentum sum rules. In the second part, we propose how to include the transverse momentum dependence for the double parton distribution functions using the extension of the Kimber-Martin-Ryskin framework previously applied to the single PDFs.

1 Introduction

Hard processes in hadron collisions are usually described by the collinear factorization. This factorization can be schematically written as follows

$$d\sigma = D_1^f(x_1, Q^2) \otimes \hat{\sigma}^{ff'}(\hat{s}, Q^2) \otimes D_1^{f'}(x_1', Q^2) + O(1/Q^2) \, , \tag{1}$$

where $D_1^f(x, Q^2)$ is the standard non-perturbative collinear PDF for the partons of type f (and simiularly for f') and $\hat{\sigma}^{ff'}(\hat{s}, Q^2)$ is the partonic cross section which can be evaluated perturbatively. The hard scale Q^2 could be a heavy quark mass, transverse energy of a jet or invariant mass of the Drell-Yan pair. The above factorization formula is appropriate for the description of inclusive cross sections for single hard processes. In the new regime of high energies, such as those probed at Tevatron and more recently at the Large Hadron Collider, multiple scatterings occur more frequently. This is related to the fact that at these high energies the parton luminosities increase rapidly with the decreasing fractions of the longitudinal momenta. For the description of the processes in which two pairs of partons scatter simultaneously in one hadronic encounter, the following formula is usually utilized

$$d\sigma = D_2^{f_1 f_2}(x_1, x_2; Q_1^2, Q_2^2; q_T) \otimes \hat{\sigma}^{f_1 f_1'}(\hat{s}_1, Q_1^2) \otimes \hat{\sigma}^{f_2 f_2'}(\hat{s}_2, Q_2^2) \otimes D_2^{f_1' f_2'}(x_1', x_2'; Q_1^2, Q_2^2; q_T) \, , \tag{2}$$

where $D_2^{f_1 f_2}(x_1, x_2; Q_1^2, Q_2^2; q_T)$ is the double collinear PDF. The formula in Eq. (2) is meant to represent the scattering of two pairs of partons, with the hard scattering described by the two hard parton

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cross sections at hard scales Q_1 and Q_2. The additional scale q_T is the relative transverse momentum of the two partons. The above formula is widely used in the phenomenology of the double parton scattering, though strictly speaking it has not yet been rigorously proven. Important steps towards the proof for the double Drell-Yan process have been provided in [1].

The single PDFs are non-perturbative quantities, but their evolution with the hard scale can be described by the DGLAP evolution equations, with the splitting functions which can be calculated perturbatively. The double PDFs also obey the evolution equations which have in the leading logarithmic approximation the following DGLAP-like form, see e.g. [2] for more details,

$$\partial_t D_{f_1 f_2}(x_1, x_2; t) = \sum_{f'} \int_0^{1-x_2} du\, \mathcal{K}_{f_1 f'}(x_1, u, t)\, D_{f' f_2}(u, x_2; t) \\ + \sum_{f'} \int_0^{1-x_1} du\, \mathcal{K}_{f_2 f'}(x_2, u, t)\, D_{f_1 f'}(x_1, u; t) + \sum_{f'} \mathcal{K}^R_{f' \to f_1 f_2}(x_1, x_2, t) D_{f'}(x_1 + x_2; t)\,. \quad (3)$$

Here, we have used the notation in which $D_{f_1 f_2}(x_1, x_2; t)$ denotes the double PDF which depends on two equal hard scales $Q_1 = Q_2 = Q$ and the evolution variable is defined as $t = \ln Q^2/Q_0^2$. Also, the above evolution equations are defined for the case of zero q_T momentum. The kernels $\mathcal{K}$ are the Altarelli-Parisi splitting functions, with $\mathcal{K}^R$ being the real part of the splitting function only. The physical interpretation of these evolution equations is as follows: the first two terms describe the situation where one of the two partons undergoes the collinear splitting, and the third term describes the situation when one parton collinearly splits into two. The first two terms are homogeneous contribution, whereas the third, splitting term, is the non-homogeneous contribution since it depends on the single parton distribution function. The above equations were first discussed in the context of the jet structure, and later on derived for the the parton distribution functions.

2 Momentum sum rules

The evolution equations (3) need to be solved simultaneously with the DGLAP equations for the single parton distirbution functions. In addition, there are sum rules which must be obeyed by the solutions to the evolution equations for both the double and single PDFs. There are known sum rules for the single parton distribution functions like the momentum sum rule and the quark number sum rule. The corresponding momentum sum rule for the double PDFs reads

$$\sum_{f_1} \int_0^{1-x_2} dx_1\, x_1\, \frac{D_{f_1 f_2}(x_1, x_2; t)}{D_{f_2}(x_2; t)} = 1 - x_2\,. \quad (4)$$

The integrand in this sum rule is an expression of the conditional probability to find the parton f_1 with the momentum fraction x_1 while keeping the second parton f_2 with momentum x_2 fixed. The total momentum fraction carried by the partons f_1 under this condition is equal to $1 - x_2$. In the similar manner, one can construct the quark number sum rule for the double PDFs [2].

The important feature of the evolution equations is that they conserve the sum rules. If the PDFs satisfy the sum rules at certain scale t_0, then they will also satisfy these sum rules at higher scale t_1 after the evolution. The presence of the splitting term in the evolution equation for the double PDFs is essential to guarantee this property.

2.1 Initial conditions

Thus for the practical purposes, in phenomenological applications one needs to impose suitable initial conditions both for the single and double PDFs simultaneously. At the same time, the momentum sum rules should be also satisfied. First attempts to resolve this problem were given in [2]. In reference [3] another solution was proposed which, however, did not have the symmetry with respect to the exchange of the partons. Another proposal was given in [4] where the construction was based on the Dirichlet distributions. The obtained solution was positive definite and symmetric with respect to the exchange of the two partons.

The framework of reference [4] rests on the observation that the natural functions which obey the sum rules for the single PDFs and double PDFs are the Dirichlet distributions. Let us consider the following form for the single and double PDFs,

$$D(x) = N_1 x^{-\alpha}(1-x)^{\beta} \ , \qquad D(x_1,x_2) = N_2 x_1^{-\tilde{\alpha}} x_2^{-\tilde{\alpha}}(1-x_1-x_2)^{\tilde{\beta}} \ . \tag{5}$$

One can rewrite them using the Mellin transform,

$$\tilde{D}(n) = \int_0^1 dx\, x^{n-1}\, D(x)\,, \qquad \tilde{D}(n_1,n_2) = \int_0^1 dx_1\, x_1^{n_1-1} \int_0^{1-x_1} dx_2\, x_2^{n_2-1}\, D(x_1,x_2)\ , \tag{6}$$

to obtain

$$\tilde{D}(n) = N_1 \frac{\Gamma(n-\alpha)\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)}, \qquad \tilde{D}(n_1,n_2) = N_2 \frac{\Gamma(n_1-\tilde{\alpha})\Gamma(n_2-\tilde{\alpha})\Gamma(\tilde{\beta}+1)}{\Gamma(n_1+n_2-2\tilde{\alpha}+\tilde{\beta}+1)}\ . \tag{7}$$

The momentum sum rule in the Mellin space can be rewritten as:

$$\tilde{D}(n_1,2) = \tilde{D}(n_1) - \tilde{D}(n_1+1)\,, \tag{8}$$

and similarily for the second parton. It can be easily seen, using some basic properties of the Beta function that the form for the single and double PDFs satisfies the above momentum sum rule. In particular the powers of the two distributions need to satisfy

$$\alpha = \tilde{\alpha}, \qquad \tilde{\beta} = \beta + \alpha - 1\,, \tag{9}$$

with the normalizations also being uniquely defined. We see therefore that the small x powers for the single and double PDFs are the same. On the other hand, the large x power of the correlating factor of the double PDF is related to the sum of the small and large x powers of the single PDF.

In practice, the parametrizations for the PDFs need to be more complicated than the single Dirichlet distribution, nevertheless the framework can be generalized to the sum of arbitrary number of such terms. In such a case, for each term in the series there would be a condition similar to Eq. (9). Thus the presented framework allows for the unique construction of the functional form of the initial conditions for the double parton distributions, using the knowledge of the functional form of the single parton distribution functions if the latter ones can be expressed in terms of the series of the Beta distributions. The powers of all the terms in the expansion for the double PDFs are uniquely fixed by the powers of the single PDFs. In the case of the single channel, the normalization for the double PDFs is also fixed in this framework.

As an example of the application we have used the MSTW2008 LO parametrization for the gluon distribution to construct the double gluon distribution D_{gg}. To illustrate the correlations in the double

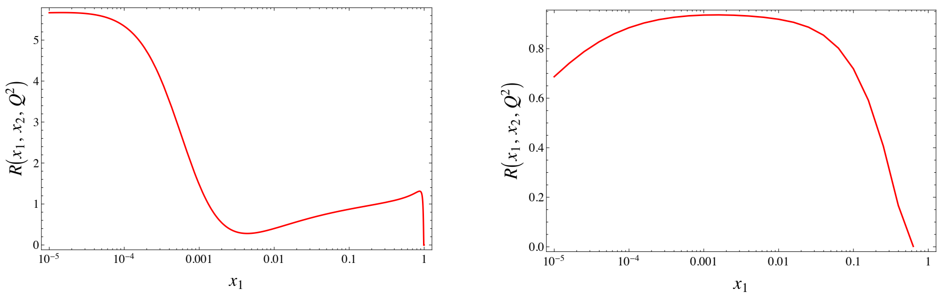

Figure 1. The ratio of the double PDF to the product of single PDFs, constructed in the framework using Dirichlet distributions from MSTW2008 LO parametrization. Left: the ratio at the initial scale $Q^2 = 1$ GeV^2 for $x_2 = 0.01$. Right: the ratio evolved to $Q^2 = 25$ GeV^2 and $x_2 = 0.03$.

distributions it is useful to present the results in the form of the ratio of the double PDF and the product of the single PDFs

$$R_{gg}(x_1, x_2; t) \;=\; \frac{D_{gg}(x_1, x_2; t)}{D_g(x_1; t) D_g(x_2; t)}\;. \tag{10}$$

In the left plot in figure 1 the ratio is showed for the initial scale $Q^2 = 1$ GeV^2. We see that for this very low scale, the ratio is significantly different than unity for wide range of x, which implies that the correlations are very large. On the right hand plot, we show the ratio after the single and double PDFs have been evolved to a higher scale using the evolution equations. We have checked explicitly that the evolution does preserve the momentum sum rule which has been imposed onto the initial conditions within the presented construction. The evolved ratio indicates that most of the correlations are washed out by the LO evolution, except at the highest value of the longitudinal momentum

3 Unintegrated double PDFs

An important aspect for the description of the processes in hadronic collisions is the transverse momentum dependence of the parton distribution functions. The collinear description has certain limitations in that it does not correctly describe the kinematics of the reaction, particularly when one considers more exclusive processes. In that context, the transverse momentum dependent (or unintegrated parton distributions) gained a lot of attention due to the fact that they encode more correct kinematics of the process. The question thus can be posed as to how to construct the unintegrated double parton distribution functions. In [5] a practical approach has been suggested which is based on the formulation originally proposed in [6], called the KMR (Kimber-Martin-Ryskin) approach for the single unintegrated PDFs. The idea behind this approach, which is accurate to the leading power, is that the unintegrated PDFs can be obtained using the integrated PDFs and the Sudakov form factors. The benefits are that one can utilize the integrated PDFs obtained from the standard DGLAP equations to introduce the transverse momentum dependence.

The construction of the KMR framework of the unintegrated PDFs starts from the DGLAP evolution equation written in the following form

$$\frac{\partial D_a(x,\mu)}{\partial \ln \mu^2} = \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z}\, P_{aa'}(z,\mu)\, D_{a'}\Big(\frac{x}{z},\mu\Big) - D_a(x,\mu) \sum_{a'} \int_0^{1-\Delta} dz z P_{a'a}(z,\mu)\,, \tag{11}$$

where the first term comes from the real emissions and the second term is the virtual contribution. The cutoff Δ is introduced to regulate the soft divergencies in both terms. After the integration of the virtual part the solution can be written in the following form

$$D_a(x,Q) = T_a(Q,Q_0,)D_a(x,Q_0) + \int_{Q_0^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2}\, f_a(x,k_\perp,Q)\,, \tag{12}$$

where the unintegrated parton distribution function (UPDF) are given by

$$f_a(x,k_\perp,Q) \equiv T_a(Q,k_\perp)\sum_{a'}\int_x^{1-\Delta}\frac{dz}{z}\,P_{aa'}(z,k_\perp)D_{a'}\Big(\frac{x}{z},k_\perp\Big)\,, \tag{13}$$

with $T_a(Q,k_\perp)$ being the Sudakov form factor. The transverse momentum dependent density is thus built from the standard integrated density in the last step of the evolution through the inclusion of the Sudakov form factor. The cutoff parameter needs to be specified in order to fully fix the unintegrated PDF. In [6] $\Delta = k_\perp/Q$ was chosen in the spirit of the DGLAP strongly ordered emissions. In the following works, for example [7], angular ordering was considered which results in $\Delta = k_\perp/(k_\perp + Q)$ and allows for the smooth transition into the region $k_\perp > Q$.

In order to extend the KMR construction to the double PDFs one can use the parton-to-parton evolution functions, which evolve the parton densities from initial scale to final scale, i.e.

$$\tilde{D}_a(n,\mu) = \sum_b \tilde{E}_{ab}(n,\mu,\mu_0)\,\tilde{D}_b(n,\mu_0)\,, \tag{14}$$

where Mellin space representation was used, with variable n, instead of the direct x space (see Eq. (6)). These parton-to-parton evolution functions $\tilde{E}$ obey the DGLAP equations, and therefore the virtual part can also be integrated out.

Then, one can represent the solution to the evolution equations for the double PDFs (3) by means of these evolution functions [8],

$$\begin{aligned}\tilde{D}_{a_1a_2}(n_1,n_2,Q_1,Q_2) = \sum_{a',a''}\Big\{&\tilde{E}_{a_1a'}(n_1,Q_1,Q_0)\,\tilde{E}_{a_2a''}(n_2,Q_2,Q_0)\,\tilde{D}_{a'a''}(n_1,n_2,Q_0,Q_0)\\ &+\int_{Q_0^2}^{Q^2_{\min}}\frac{dQ_s^2}{Q_s^2}\,\tilde{E}_{a_1a'}(n_1,Q_1,Q_s)\,\tilde{E}_{a_2a''}(n_2,Q_2,Q_s)\,\tilde{D}^{(sp)}_{a'a''}(n_1,n_2,Q_s)\Big\}\,,\end{aligned} \tag{15}$$

where $Q^2_{\min} = \min\{Q_1^2, Q_2^2\}$ and $\tilde{D}^{(sp)}_{a'a''}$ is the non-homogeneous contribution originating from the single parton density. Using the solutions for the parton-to-parton evolution functions one can perform the same substitution as was done for the case of the single PDFs and find the corresponding unintegrated double parton distribution function. It turns out however, that in that case there are three distinct regions of momenta, where different forms of the unintegrated double PDFs are obtained:

For the case when $k_{1\perp} \leq Q_0$ and $k_{2\perp} > Q_0$, we have

$$\tilde{f}^{(h)}_{a_1a_2}(n_1,n_2,k_{2\perp},Q_1,Q_2) = T_{a_1}(Q_1,Q_0)\,T_{a_2}(Q_2,k_{2\perp})\sum_b \tilde{P}_{a_2b}(n_2,k_{2\perp})\,\tilde{D}^{(h)}_{a_1b}(n_1,n_2,Q_0,k_{2\perp})\,. \tag{16}$$

The dependence of the transverse momentum $k_{1\perp}$ is integrated over up to Q_0 in such a case and $k_{1\perp}$ is not present among the arguments of the defined function. The effect of such an integration is hidden in the integrated double PDFs on the r.h.s. taken at the scale Q_0 for the first parton.

Similarly, for $k_{1\perp} > Q_0$ and $k_{2\perp} \le Q_0$, we have

$$\tilde{f}^{(h)}_{a_1a_2}(n_1,n_2,k_{1\perp},Q_1,Q_2) = T_{a_1}(Q_1,k_{1\perp})\, T_{a_2}(Q_2,Q_0) \sum_b \tilde{P}_{a_1b}(n_1,k_{1\perp})\, \tilde{D}^{(h)}_{ba_2}(n_1,n_2,k_{1\perp},Q_0)\,. \quad (17)$$

Now the momentum $k_{2\perp}$ is integrated up to the scale Q_0 and only $k_{1\perp}$ dependence is present. Finally, for $k_{1\perp}, k_{2\perp} > Q_0$ the unintegrated double PDF has the form

$$\begin{aligned}\tilde{f}^{(h)}_{a_1a_2}(n_1,n_2,k_{1\perp},k_{2\perp},Q_1,Q_2) &= T_{a_1}(Q_1,k_{1\perp})\, T_{a_2}(Q_2,k_{2\perp}) \\ &\times \sum_{b,c} \tilde{P}_{a_1b}(n_1,k_{1\perp})\, \tilde{P}_{a_2c}(n_2,k_{2\perp}) \tilde{D}^{(h)}_{bc}(n_1,n_2,k_{1\perp},k_{2\perp})\,. \end{aligned} \quad (18)$$

The treatment of the non-homogeneous term is much more complicated because there are two potential contributions to the transverse momentum dependence. One can have the transverse momentum either from the evolution of one or both partons after the splitting or from the splitting vertex itself. In the first case, it is sufficient to consider the inhomogeneous term from formula (15) and use the solution for the parton evolution functions as we have done before. For the second contribution, one needs to compute the exact form of the vertex including the transverse momentum dependence, and therefore treat the splitting beyond the collinear approximation.

Summary

In this presentation we have discussed two issues related with the double parton distribution functions. In the first part, we have proposed the framework for the construction of the initial conditions for the double PDFs using the information from the single PDFs and the constraints from the sum rules. It turns out that (in the single channel case) there is a unique solutions, which allows for the construction of such initial conditions. In the second part, we have proposed a generalization of the KMR framework for the unintegrated PDFs to the double PDFs. The treatment of the homogeneous term is relatively straightforward and one arrives at three different perturbative contributions for the unintegrated double PDFs depending on the relevant hierarchy of hard scales. The treatment of the non-homogeneous term is more complicated as it involves two sources of the transverse momenta either from the evolution or from the splitting vertex itself, and therefore in principle it goes beyond the collinear approximation. The presented framework for unintegrated DPDFs relies on the use of the integrated double PDFs and can be implemented numerically.

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