Evolution equations for double parton distributions

Initial conditions and transverse momentum dependence

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Outline

- Evolution equations for double Parton Distribution Functions (dPDFs)
- Sum rules
- Initial conditions
- Examples for the single channel: gluons
- Unintegrated dPDFs
- Summary and outlook

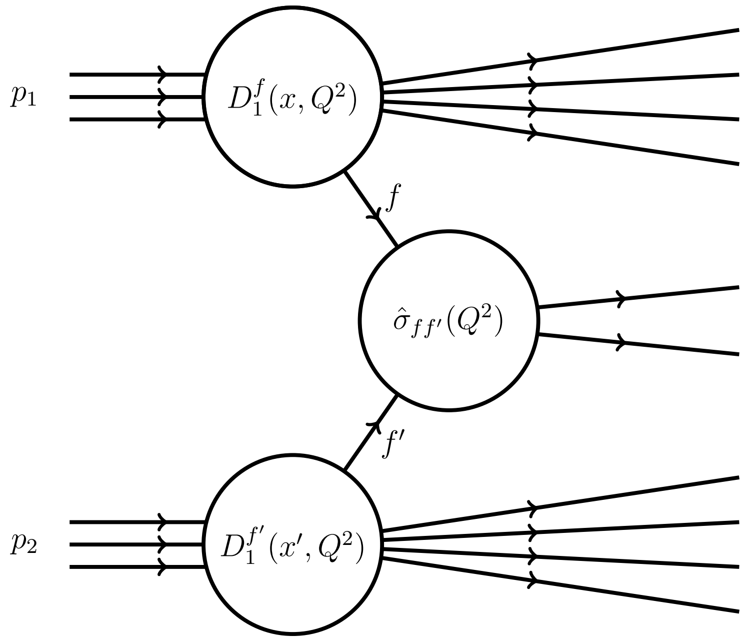
References:

Initial conditions: *Golec-Biernat, Lewandowska, Serino, Snyder, AS, arXiv: 1507.08583, Phys.Lett. B750 (2015) 559-564*

Unintegrated dPDFs: *Golec-Biernat, AS, work in progress*

Single scattering process

Single parton scattering: one hard process



Single collinear PDF:

$$D_1^f(x,Q^2)$$

Partonic cross section:

$$\hat{\sigma}^{ff'}(\hat{s},Q^2,m^2)$$

Collins, Soper, Sterman

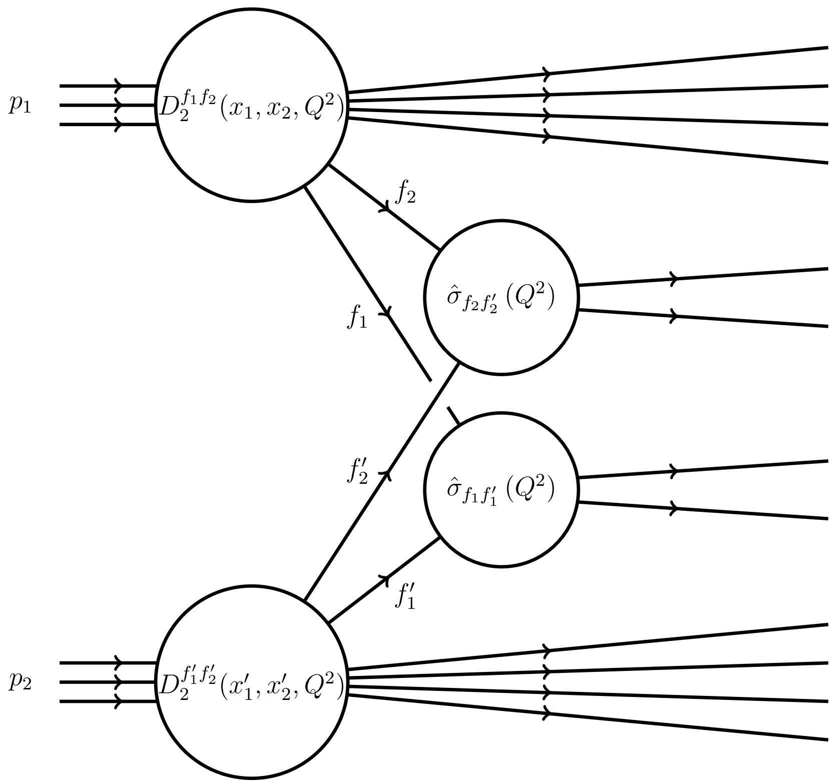
Collinear factorization:

Given the presence of the hard scale, the cross section (up to power corrections) can be factorized into perturbatively calculable partonic cross section and non-perturbative parton distribution functions.

$$d\sigma = D_1^f(x_1,Q^2) \otimes \hat{\sigma}^{ff'}(\hat{s},Q^2,m^2) \otimes D_1^{f'}(x_2,Q^2) + \mathcal{O}(\frac{1}{Q^2})$$

Double scattering process

Double parton scattering: two hard processes



Two types of partons: f_1, f_2

Two momentum fractions: x_1, x_2 $\quad x_1 + x_2 \leq 1$

Two hard scales: $Q_1, Q_2 \gg \Lambda_{QCD}$

Relative transverse momentum: q_T

Double PDF (DPDF): $D_2^{f_1, f_2}(x_1, x_2, Q_1^2, Q_2^2; q_T)$

Factorization formula(?):

$$d\sigma = D_2^{f_1 f_2}(x_1, x_2; Q_1^2, Q_2^2) \otimes \hat{\sigma}^{f_1 f_1'}(\hat{s}_1, Q_1^2)$$
$$\otimes \hat{\sigma}^{f_2 f_2'}(\hat{s}_2, Q_2^2) \otimes D_2^{f_1' f_2'}(x_1', x_2'; Q_1^2, Q_2^2)$$

Diehl,Gaunt,Ostermeier,Ploessl,Schaefer Important steps towards the proof in double Drell-Yan process.

Evolution equations for single PDFs

DGLAP evolution equation for single PDF:

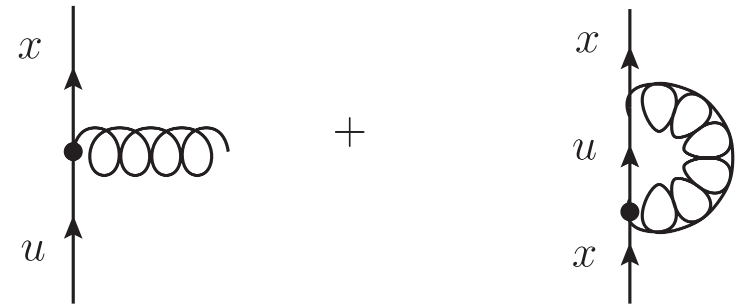
$$\partial_t D_f(x,t) = \sum_{f'} \int_0^1 du\, \mathcal{K}_{ff'}(x,u,t)\, D_{f'}(u,t)$$

Evolution variable:

$$t = \ln Q^2/Q_0^2$$

Real and virtual parts of the kernel:

$$\mathcal{K}_{ff'}(x,u,t) = \mathcal{K}^R_{ff'}(x,u,t) - \delta(u-x)\,\delta_{ff'}\,\mathcal{K}^V_f(x,t)$$



Real emission kernel:

$$\mathcal{K}^R_{ff'}(x,u,t) = \frac{1}{u} P_{ff'}(\frac{x}{u},t)\,\theta(u-x)$$

Splitting functions:

$$P_{ff'}(z,t) = \frac{\alpha_s(t)}{2\pi} P^{(0)}_{ff'}(z) + \frac{\alpha_s^2(t)}{(2\pi)^2} P^{(1)}_{ff'}(z) + \dots .$$

Evolution equations for single PDFs

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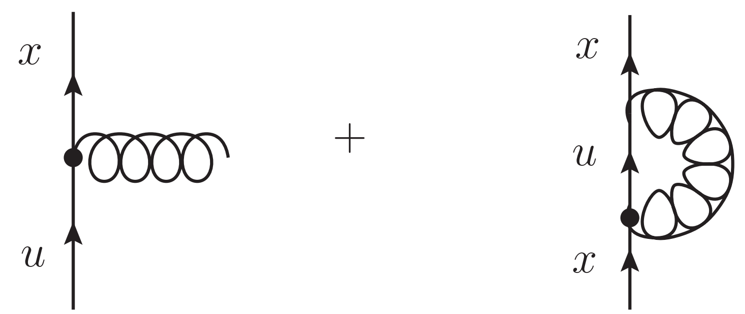
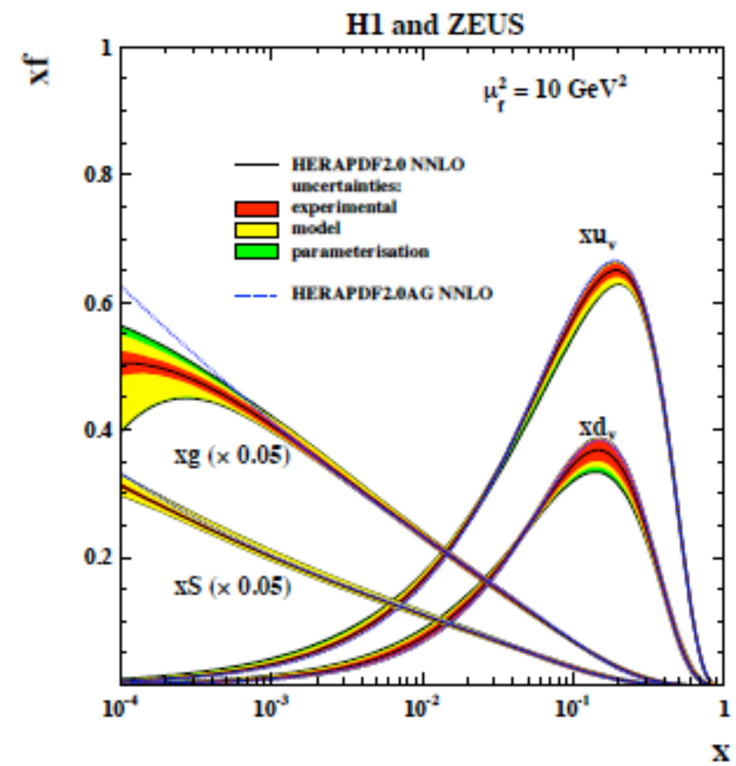


Fits and extraction(see talk by *Voica Radescu*)



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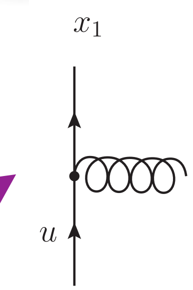
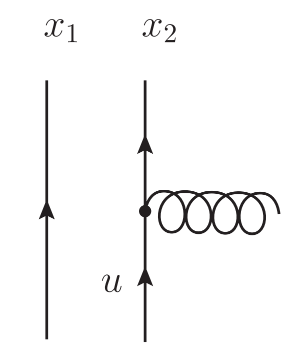
Flexible initial conditions, constrained by the momentum and sum rule only.

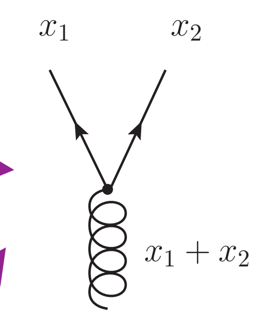
Evolution equations for double PDFs

DGLAP evolution equation for double PDF:

$$
\begin{aligned}
\partial_t D_{f_1 f_2}(x_1,x_2,t) &= \sum_{f'} \int_0^{1-x_2} du\, \mathcal{K}_{f_1 f'}(x_1,u,t)\, D_{f' f_2}(u,x_2,t) \\
&+ \sum_{f'} \int_0^{1-x_1} du\, \mathcal{K}_{f_2 f'}(x_2,u,t)\, D_{f_1 f'}(x_1,u,t) \\
&+ \sum_{f'} \mathcal{K}^R_{f' \to f_1 f_2}(x_1,x_2,t)\, D_{f'}(x_1+x_2,t)
\end{aligned}
$$







Konishi, Ukawa, Veneziano; Snigirev, Zinovev, Shelest

Inhomogeneous term

Splitting term of one parton into two:

$$
\mathcal{K}^R_{f' \to f_1 f_2}(x_1,x_2,t) = \frac{\alpha_s(t)}{2\pi} \frac{1}{x_1+x_2} P^{(0)}_{f' f_1}\left(\frac{x_1}{x_1+x_2}\right)
$$

Evolution equation for double PDFs is coupled with single PDFs.

Need to be solved together with suitable initial conditions.

Sum rules for single and double PDFs

Momentum sum rule for single PDFs

$$\sum_f \int_0^1 dx\, x D_f(x,t) = 1$$

Quark number sum rule for single PDFs

$$\int_0^1 dx\, \{D_{q_i}(x,t) - D_{\bar{q}_i}(x,t)\} = N_i$$

Momentum sum rule for double PDFs

$$\sum_{f_1} \int_0^{1-x_2} dx_1 x_1 \frac{D_{f_1 f_2}(x_1,x_2,t)}{D_{f_2}(x_2,t)} = 1 - x_2$$

Conditional probability to find the parton f_1 with the momentum fraction x_1 while keeping fixed the second parton f_2 with momentum x_2 .

Valence quark number sum rule for double PDFs

$$\int_0^{1-x_2} dx_1 \{D_{q_i f_2}(x_1,x_2,t) - D_{\bar{q}_i f_2}(x_1,x_2,t)\}$$
$$= \begin{cases} N_i\, D_{f_2}(x_2,t) & \text{for } f_2 \neq q_i, \bar{q}_i \\ (N_i - 1)\, D_{f_2}(x_2,t) & \text{for } f_2 = q_i \\ (N_i + 1)\, D_{f_2}(x_2,t) & \text{for } f_2 = \bar{q}_i \end{cases}$$

If sum rules hold for initial conditions they will hold for higher scales after the evolution.

How to consistently impose the initial conditions for sPDF and dPDF with sum rules?

Problem of initial conditions in dPDFs

Usually simplifying assumption is taken:

$$D_{f_1 f_2}(x_1, x_2) = D_{f_1}(x_1) D_{f_2}(x_2)$$

Factorizable ansatz, could work well for rather small x but is inconsistent with sum rules.

Improvement with correlating factor:

Gaunt, Stirling

$$D_{f_1 f_2}(x_1, x_2) = D_{f_1}(x_1) D_{f_2}(x_2) \frac{(1 - x_1 - x_2)^2}{(1 - x_1)^{2+n_1}(1 - x_2)^{2+n_2}}$$

Takes into account some correlation but still does not obey sum rules exactly.

Initial conditions: Dirichlet distribution

Consider Beta distribution and gluons only (for now)

$$D(x) = N_1 \, x^{-\alpha} \, (1-x)^{\beta}$$

Mellin transform:

$$\tilde{D}(n) = \int_0^1 dx x^{n-1} D(x)$$

Momentum sum rule in Mellin space:

$$\tilde{D}(2) = 1$$

$$\tilde{D}(n) \; = \; \frac{1}{B(2-\alpha, 1+\beta)} \int_0^1 dx \, x^{n-1} x^{-\alpha} (1-x)^{\beta} = \frac{B(n-\alpha, \beta+1)}{B(2-\alpha, \beta+1)}$$

Take the ansatz for double distribution in the form of the Dirichlet distribution:

$$D(x_1, x_2) = N_2 \, x_1^{-\tilde{\alpha}} x_2^{-\tilde{\alpha}} (1 - x_1 - x_2)^{\tilde{\beta}}$$

Double Mellin transform:

$$\tilde{D}(n_1, n_2) = \int_0^1 dx_1 x_1^{n_1 - 1} \int_0^1 dx_2 x_2^{n_2 - 1} D(x_1, x_2) \longrightarrow \tilde{D}(n_1, n_2) = N_2 \frac{\Gamma(n_1 - \tilde{\alpha}) \Gamma(n_2 - \tilde{\alpha}) \Gamma(1 + \tilde{\beta})}{\Gamma(n_1 + n_2 + 1 + \tilde{\beta} - 2\tilde{\alpha})}$$

Initial conditions: relating the parameters

The momentum sum rule for dPDFs in Mellin space

LHS: Double PDFs in Mellin space →

$$\tilde{D}(n_1,2) = \tilde{D}(n_1) - \tilde{D}(n_1+1)$$

$$\tilde{D}(2,n_2) = \tilde{D}(n_2) - \tilde{D}(n_2+1)$$

← RHS: Single PDFs in Mellin space

RHS:

$$\tilde{D}(n_1) - \tilde{D}(n_1+1) = \frac{1}{B(2-\alpha,\beta+1)}\big(B(n_1-\alpha,\beta+1) - B(n_1+1-\alpha,\beta+1)\big) = \frac{1}{B(2-\alpha,\beta+1)}\frac{\Gamma(n_1-\alpha)\Gamma(2+\beta)}{\Gamma(2+\beta+n_1-\alpha)}$$

Where the following property of Beta function was used:

$$B(a,b) = B(a+1,b) + B(a,b+1)$$

LHS:

$$\tilde{D}(n_1,2) = N_2\frac{\Gamma(n_1-\tilde{\alpha})\Gamma(2-\tilde{\alpha})\Gamma(1+\tilde{\beta})}{\Gamma(n_1+3+\tilde{\beta}-2\tilde{\alpha})}$$

Comparing the functional form of both sides we see that the equality can be satisfied if

$$\tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta + \alpha - 1$$

and

$$N_2 = \frac{1}{B(2-\alpha,\alpha+\beta)B(2-\alpha,\beta+1)}$$

Initial conditions

If the single distribution is given by a Beta distribution

$$D(x) = N_1\, x^{-\alpha}\, (1-x)^{\beta}$$

There is a unique solution in terms of the Dirichlet distribution for the double parton density:

$$D(x_1,x_2) = N_2\, x_1^{-\tilde{\alpha}} x_2^{-\tilde{\alpha}} (1-x_1-x_2)^{\tilde{\beta}}$$

With powers of the dPDF being related to the powers of sPDF

$$\tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta + \alpha - 1$$

Normalization for dPDF in this particular case is uniquely determined.

Small x powers for single and double PDFs are the same.

The large x power of the correlating factor in dPDF is related to the sum of large and small x powers of the single distribution.

Initial conditions: expansion

Realistic parametrizations are however more complicated than a single Beta distribution.

Example MSTW2008 gluon PDF: $xD_1^g\left(x,Q^2\right) = N_1 x^{-\delta_g}\left(1-x\right)^{\eta_g}\left(1+\epsilon_g\sqrt{x}+\gamma_g x\right),$

However, this parametrization is sum of Beta distributions of the form:

$$D(x) = N_1 \sum_{k=1}^{K} a_k x^{-\alpha_k}\left(1-x\right)^{\beta_k}$$

Assuming that the dPDF is the sum of Dirichlet distributions:

$$D(x_1,x_2) = N_2 \sum_{k=1}^{K} c_k x_1^{-\tilde{\alpha}_k} x_2^{-\tilde{\alpha}_k}\left(1-x_1-x_2\right)^{\tilde{\beta}_k}$$

Performing the same analysis as before (for single channel) one obtains the conditions for each k:

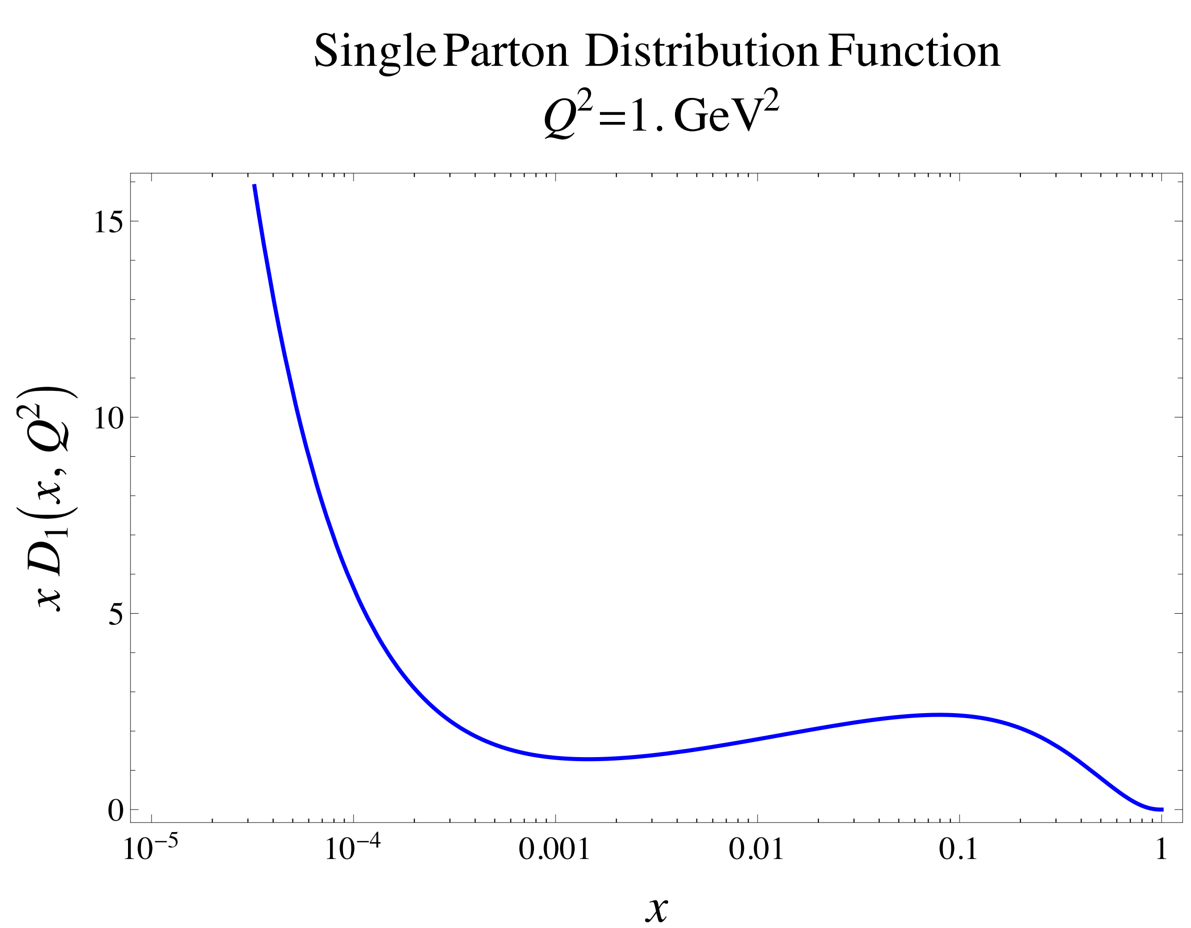
$$\tilde{\alpha}_k = \alpha_k \qquad \tilde{\beta}_k = \beta_k - 1 + \alpha_k$$

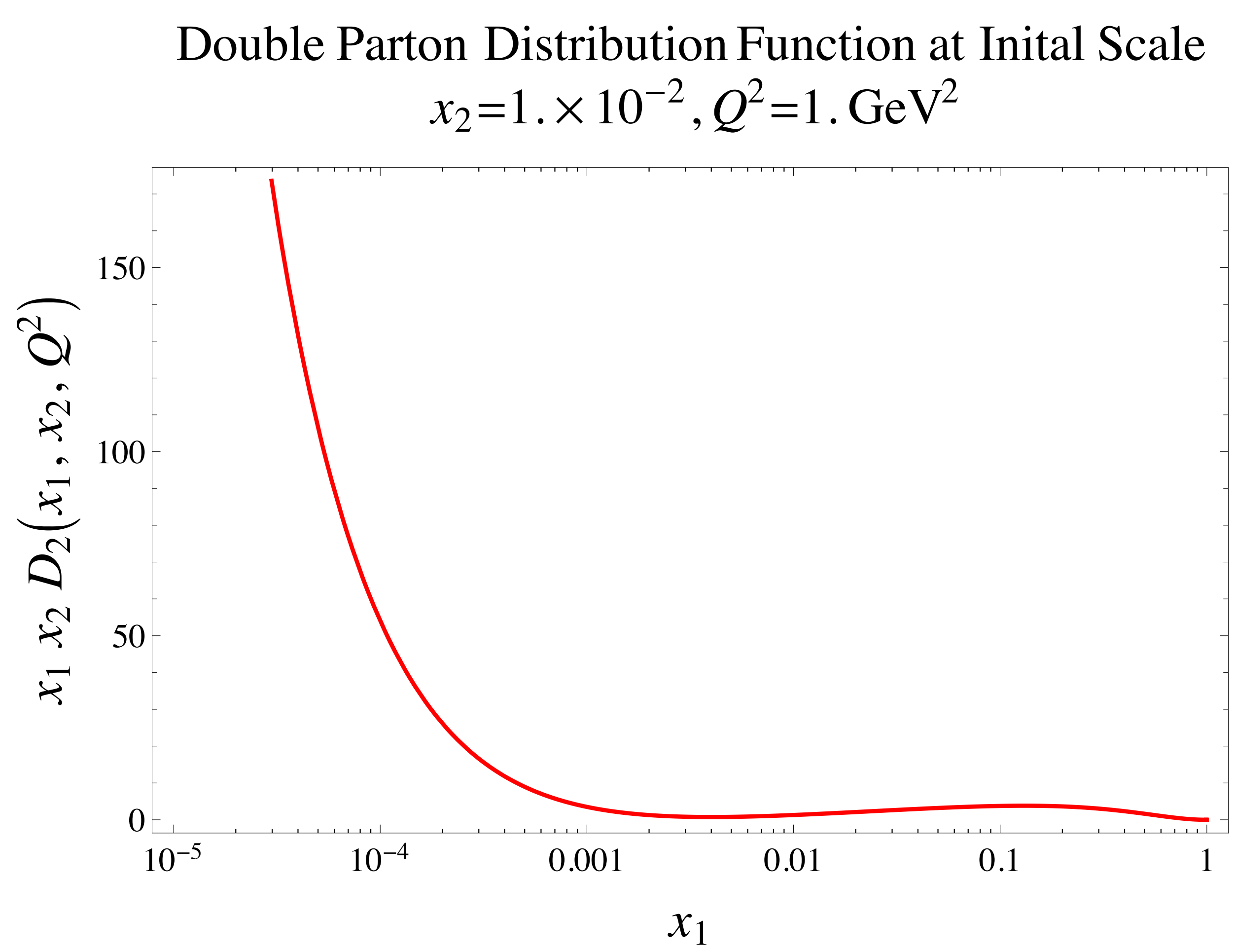
The normalizations:

$$c_k = a_k \frac{B(\alpha_1+\beta_1, 2-\alpha_1)}{B(\beta_k+\alpha_k, 2-\alpha_k)} \qquad N_2 = N_1 \frac{1}{B(\alpha_1+\beta_1, 2-\alpha_1)}$$

Initial conditions for dPDFs

- Use this algorithm, expansion in terms of Beta and Dirichlet distributions, to construct dPDF from MSTW2008 gluon.
- Single channel (gluons) only.
- Using different normalization for the LO MSTW2008 gluon.



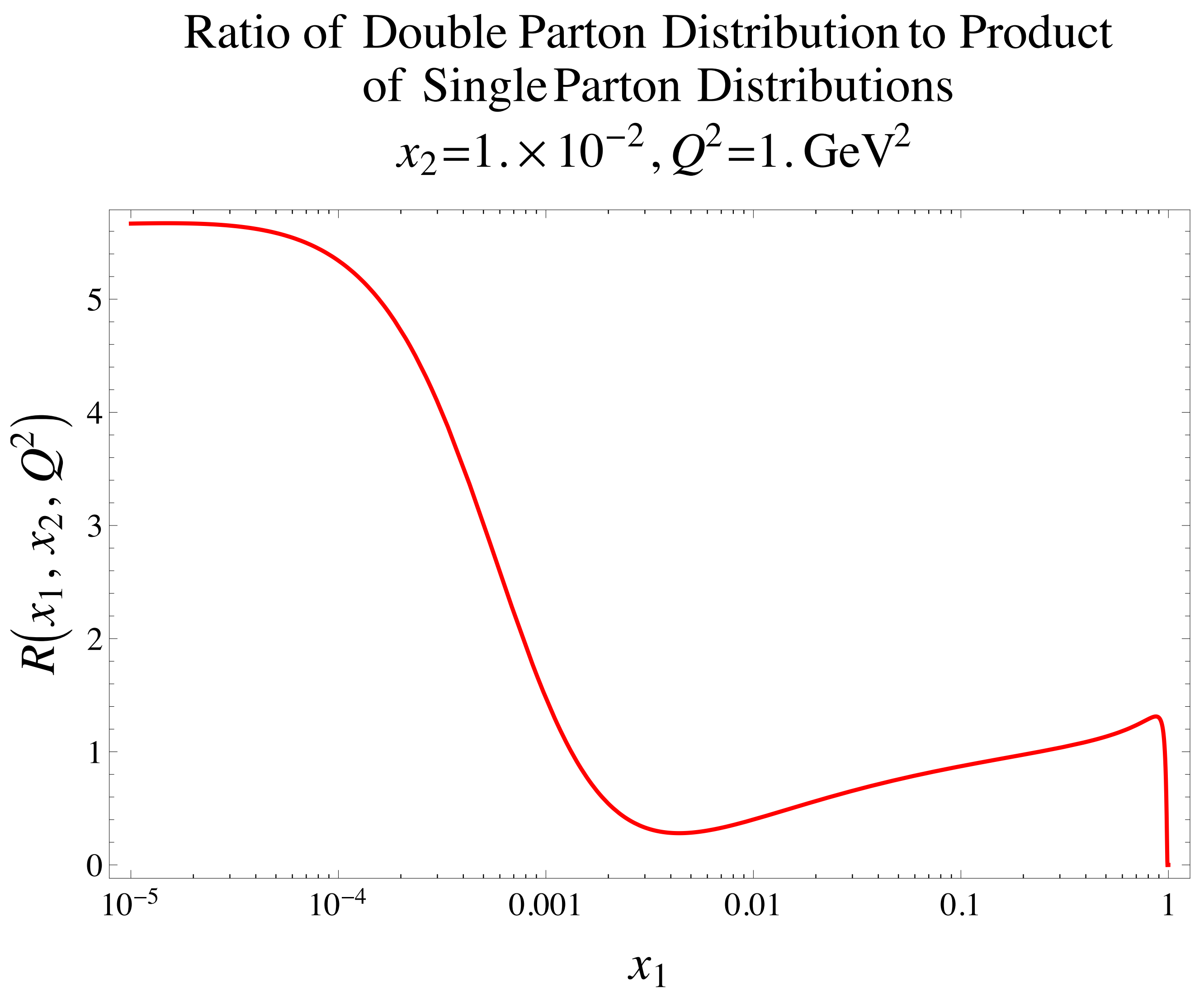


Initial conditions for dPDFs: ratios

Ratio of double distribution to product of single distributions:

$$R^{gg}\left(x_1, x_2, Q^2\right) = \frac{D_2^{gg}\left(x_1, x_2, Q^2\right)}{D_1^g\left(x_1, Q^2\right) D_1^g\left(x_2, Q^2\right)}$$

- Measure of the correlations at the initial scale.
- For this parametrization the correlations are very significant.
- Ratio different from unity over wide range of x.
- Factorization of powers at small x but different normalization.
- In principle can extend to quarks, requires some constraints put onto the form of the single PDFs.



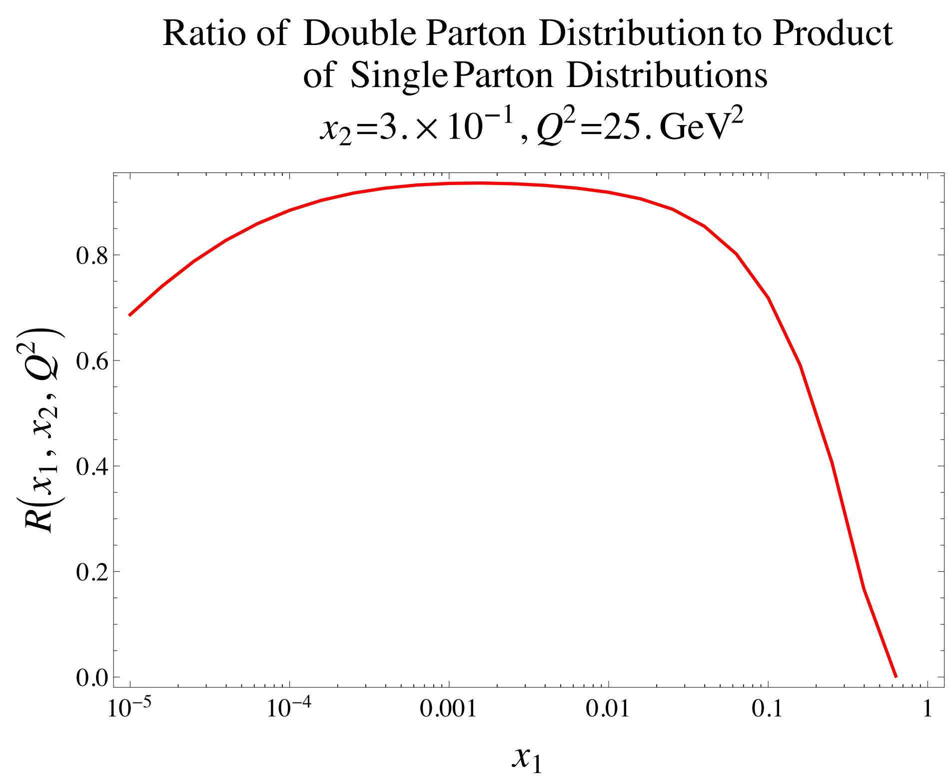
Evolution of single and double PDFs

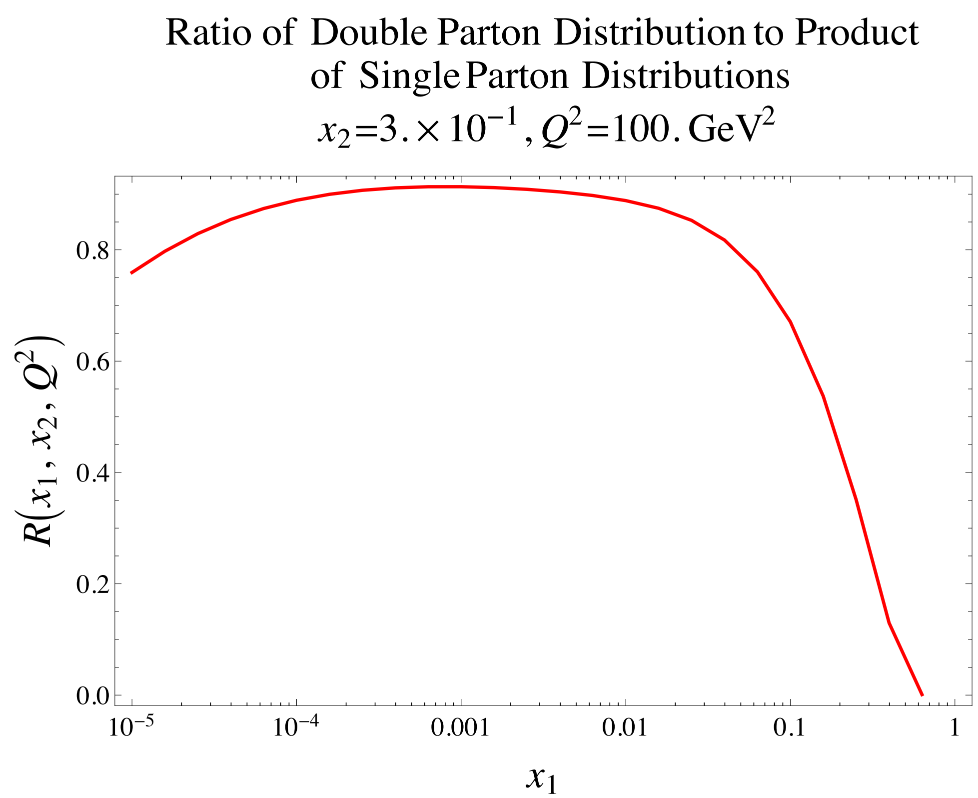
Evolve the dPDFs and sPDFs using DGLAP equations;

$$D_1^f(x, Q_0) \rightarrow D_1^f(x, Q)$$

$$D_2^{f_1 f_2}(x_1, x_2, Q_0) \rightarrow D_2^{f_1 f_2}(x_1, x_2, Q)$$

Solution found in the Mellin space and then numerically inverted to the momentum space.





Correlation washed out by evolution except for large x.

Unintegrated DPDFs

What about the transverse momentum dependence of the DPDFs?

Possible formulation:

Small x Color Glass Condensate formalism: higher Wilson line correlators

Relation to TMD (see next talk by *Daniel Boer*)

Advantages:

- consistent formulation within the small x framework
- evolution equations in principle are available (up to NLLx)

Disadvantages:

equations are rather complicated to solve for higher point correlators

Can one formulate something more practical?

- *Kimber - Martin - Ryskin* approach to the unintegrated parton densities.
- Includes transverse momentum dependence in the parton densities.
- Practical approach for the phenomenology, using integrated densities, convoluted with the Sudakov form factors

Unintegrated PDFs

Martin, Kimber, Ryskin

DGLAP evolution for single PDF

$$\frac{\partial D_a(x,\mu)}{\partial \ln \mu^2} = \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z} P_{aa'}(z,\mu)\, D_{a'}\left(\frac{x}{z},\mu\right) - D_a(x,\mu) \sum_{a'} \int_0^{1-\Delta} dz z P_{a'a}(z,\mu)$$

real virtual

after integrating out the virtual part

$$D_a(x,Q) = T_a(Q,Q_0,)D_a(x,Q_0) + \int_{Q_0^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} f_a(x,k_\perp,Q)$$

where the "unintegrated density":

$$f_a(x,k_\perp,Q) \equiv T_a(Q,k_\perp) \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z} P_{aa'}(z,k_\perp)\, D_{a'}\left(\frac{x}{z},k_\perp\right)$$

or

$$f_a(x,k_\perp,Q) = \frac{\partial}{\partial \ln k_\perp^2} \left[T_a(Q,k_\perp) D_a(x,k_\perp)\right]$$

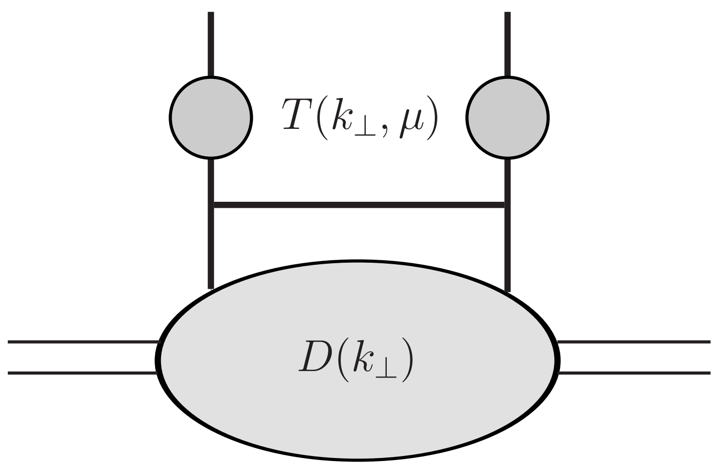
with Sudakov formfactor

$$T_a(Q,k_\perp) = \exp\left\{ - \int_{k_\perp^2}^{Q^2} \frac{dp_\perp^2}{p_\perp^2} \sum_{a'} \int_0^{1-\Delta} dz z P_{a'a}(z,p_\perp) \right\}$$

$$T_a(Q,k_\perp) \simeq 1, \quad Q \sim k_\perp \qquad T_a(Q,k_\perp) \simeq 0, \quad Q \gg k_\perp$$

Unintegrated PDFs

$$f_a(x,k_\perp,Q) \equiv T_a(Q,k_\perp) \sum_{a'} \int_x^{1-\Delta} \frac{dz}{z} P_{aa'}(z,k_\perp)\, D_{a'}\left(\frac{x}{z},k_\perp\right)$$



Dependence on two scales obtained in the last step of the evolution

Need to specify the cutoff :

DGLAP ordering: $\Delta = \frac{k_\perp}{Q}$

CCFM angular ordering: $\Delta = \frac{k_\perp}{k_\perp + Q}$

$$\Theta(\theta - \theta') \;\Rightarrow\; \mu > z k_t/(1-z) \qquad z_{max} = \frac{\mu}{\mu + k_t}$$

Larger phase space for emissions, tail in transverse momentum extends to $k_\perp > Q$

Extending the KMR framework to DPDFs

Use parton-to-parton evolution function:

(n is Mellin variable conjugated to x)

$$\tilde{D}_a(n,\mu) = \sum_b \tilde{E}_{ab}(n,\mu,\mu_0)\,\tilde{D}_b(n,\mu_0).$$

It evolves sPDF to scale μ from scale μ_0

$$\frac{\partial}{\partial \ln \mu^2}\tilde{E}_{ab}(n,\mu,\mu_0) = \sum_{a'} \tilde{P}_{aa'}(n,\mu)\,\tilde{E}_{a'b}(n,\mu,\mu_0) - \tilde{E}_{ab}(n,\mu,\mu_0)\sum_{a'}\int_0^1 dz z P_{a'a}(z,\mu)$$

initial condition
$\tilde{E}_{ab}(n,\mu_0,\mu_0) = \delta_{ab}.$

Formally integrating out virtual part:

$$\tilde{E}_{ab}(n,Q,Q_0) = T_a(Q,Q_0)\,\delta_{ab} + \int_{Q_0^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}T_a(Q,k_\perp)\sum_{a'}\tilde{P}_{aa'}(n,k_\perp)\,\tilde{E}_{a'b}(n,k_\perp,Q_0)$$

Double parton distributions (DGLAP eq):

$$\tilde{D}_{a_1a_2}(n_1,n_2,\mu_1,\mu_2) = \sum_{a',a''}\Big\{\tilde{E}_{a_1a'}(n_1,\mu_1,\mu_0)\,\tilde{E}_{a_2a''}(n_2,\mu_2,\mu_0)\,\tilde{D}_{a'a''}(n_1,n_2,\mu_0,\mu_0)$$

homogenous term

$$+\int_{\mu_0^2}^{\mu_{min}^2}\frac{d\mu_s^2}{\mu_s^2}\tilde{E}_{a_1a'}(n_1,\mu_1,\mu_s)\,\tilde{E}_{a_2a''}(n_2,\mu_2,\mu_s)\,\tilde{D}^{(sp)}_{a'a''}(n_1,n_2,\mu_s)\Big\}$$

inhomogenous term

Homogeneous part of DPDF evolution

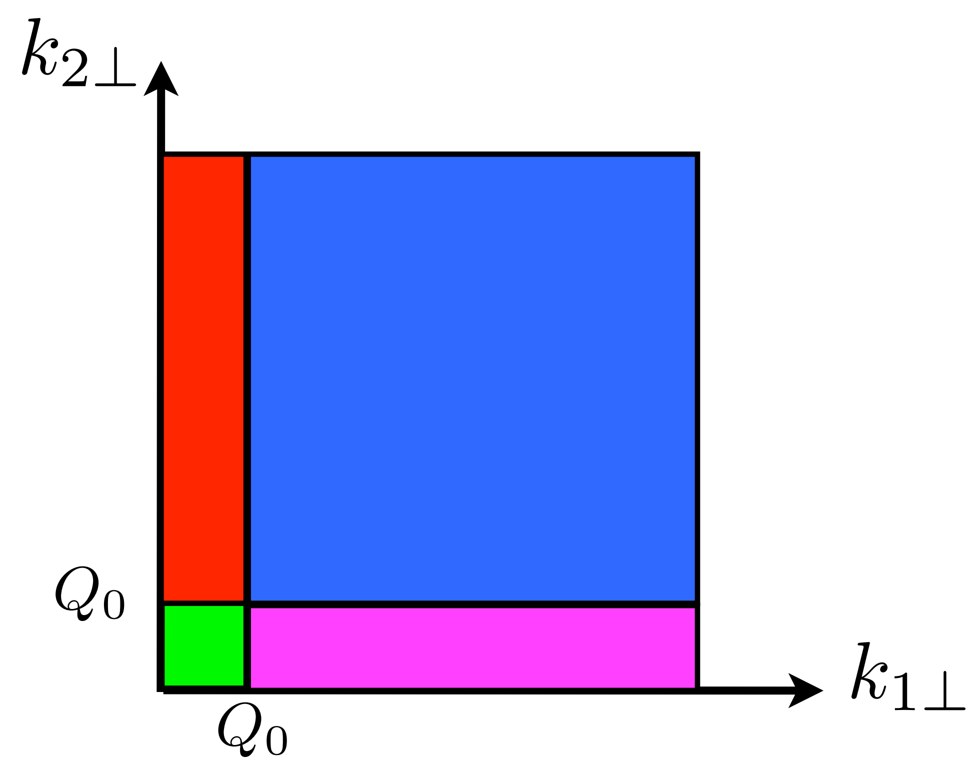
$$\tilde{D}^{(h)}_{a_1a_2}(n_1,n_2,Q_1,Q_2) = T_{a_1}(Q_1,Q_0)\,T_{a_2}(Q_2,Q_0)\,\tilde{D}_{a_1a_2}(n_1,n_2,Q_0,Q_0)$$

$$+\int_{Q_0^2}^{Q_2^2}\frac{dk_{2\perp}^2}{k_{2\perp}^2}\Big\{T_{a_1}(Q_1,Q_0)\,T_{a_2}(Q_2,k_{2\perp})\sum_b \tilde{P}_{a_2b}(n_2,k_{2\perp})\Big[\sum_{a''}\tilde{E}_{ba''}(n_2,k_{2\perp},Q_0)\tilde{D}_{a_1a''}(n_1,n_2,Q_0,Q_0)\Big]\Big\}$$

$$+\int_{Q_0^2}^{Q_1^2}\frac{dk_{1\perp}^2}{k_{1\perp}^2}\Big\{T_{a_1}(Q_1,k_{1\perp})\,T_{a_2}(Q_2,Q_0)\sum_b \tilde{P}_{a_1b}(n_1,k_{1\perp})\Big[\sum_{a'}\tilde{E}_{ba'}(n_1,k_{1\perp},Q_0)\tilde{D}_{a'a_2}(n_1,n_2,Q_0,Q_0)\Big]\Big\}$$

$$+\int_{Q_0^2}^{Q_1^2}\frac{dk_{1\perp}^2}{k_{1\perp}^2}\int_{Q_0^2}^{Q_2^2}\frac{dk_{2\perp}^2}{k_{2\perp}^2}\Big\{T_{a_1}(Q_1,k_{1\perp})\,T_{a_2}(Q_2,k_{2\perp})\sum_{b,c}\tilde{P}_{a_1b}(n_1,k_{1\perp})\,\tilde{P}_{a_2c}(n_2,k_{2\perp})\Big[\sum_{a',a''}\tilde{E}_{ba'}(n_1,k_{1\perp},Q_0)\tilde{E}_{ca''}(n_2,k_{2\perp},Q_0)\tilde{D}_{a'a''}(n_1,n_2,Q_0,Q_0)\Big]\Big\}$$

Four distinct regions of phase space depending on the ordering of scales.



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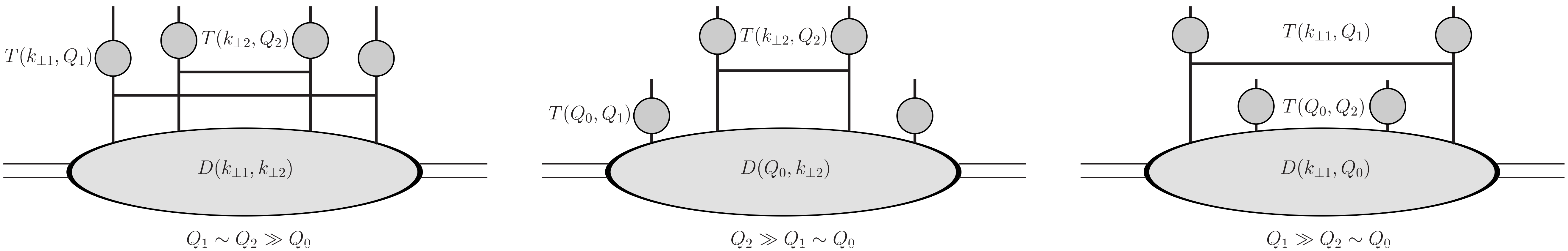
Homogeneous part



$Q_1^2 \sim Q_2^2 \gg Q_0^2$ $\quad k_{1\perp}$ unintegrated $\quad k_{2\perp}$ unintegrated

$$f^{(h)}_{a_1 a_2}(x_1, x_2, k_{1\perp}, k_{2\perp}, Q_1, Q_2) = T_{a_1}(Q_1, k_{1\perp})\, T_{a_2}(Q_2, k_{2\perp}) \sum_{b,c} \int_{\frac{x_1}{1-x_2}}^{1-\Delta_1} \frac{dz_1}{z_1} \int_{\frac{x_2}{1-x_1/z_1}}^{1-\Delta_2} \frac{dz_2}{z_2} P_{a_1 b}(z_1, k_{1\perp}) P_{a_2 c}(z_2, k_{2\perp})\, D^{(h)}_{bc}\left(\frac{x_1}{z_1}, \frac{x_2}{z_2}, k_{1\perp}, k_{2\perp}\right)$$

$Q_1^2 \sim Q_0^2$ and $Q_2^2 \gg Q_0^2$ $\quad k_{1\perp}$ integrated $\quad k_{2\perp}$ unintegrated

$$f1^{(h)}_{a_1, a_2}(x_1, x_2, k_{2\perp}, Q_1, Q_2) = T_{a_1}(Q_1, Q_0)\, T_{a_2}(Q_2, k_{2\perp}) \sum_{b} \int_{\frac{x_2}{1-x_1}}^{1-\Delta_2} \frac{dz_2}{z_2} P_{a_2 b}(z_2, k_{2\perp})\, D^{(h)}_{a_1 b}\left(x_1, \frac{x_2}{z_2}, Q_0, k_{2\perp}\right)$$

$Q_1^2 \gg Q_0^2$ and $Q_2^2 \sim Q_0^2$ $\quad k_{2\perp}$ integrated $\quad k_{1\perp}$ unintegrated

$$f1^{(h)}_{a_1, a_2}(x_1, x_2, k_{1\perp}, Q_1, Q_2) = T_{a_1}(Q_1, k_{1\perp})\, T_{a_2}(Q_2, Q_0) \sum_{b} \int_{\frac{x_1}{1-x_2}}^{1-\Delta_1} \frac{dz_1}{z_1} P_{a_1 b}(z_1, k_{1\perp})\, D^{(h)}_{b a_2}\left(\frac{x_1}{z_1}, x_2, k_{1\perp}, Q_0\right)$$

$Q_1^2 \sim Q_2^2 \sim Q_0^2$ $\quad$ non-perturbative region - parametrized by integrated density

Non-homogeneous part

In principle the same method can be applied to the non-homogeneous term:

$$\tilde{D}^{(nh)}_{a_1 a_2}(n_1, n_2, Q_1, Q_2) = \int_{Q_0^2}^{Q^2_{min}} \frac{d\mu_s^2}{\mu_s^2} \sum_{a', a''} \tilde{E}_{a_1 a'}(n_1, Q_1, \mu_s)\, \tilde{E}_{a_2 a''}(n_2, Q_2, \mu_s)\, \tilde{D}^{(sp)}_{a' a''}(n_1, n_2, \mu_s)$$

Plugging in the expressions for parton-to-parton evolution one obtains:

$$\tilde{D}^{(nh)}_{a_1 a_2}(n_1, n_2, Q_1, Q_2) = \int_{Q_0^2}^{Q^2_{min}} \frac{d\mu_s^2}{\mu_s^2} \Big[T_{a_1}(Q_1, \mu_s)\, T_{a_2}(Q_2, \mu_s)\, \tilde{D}^{(sp)}_{a_1 a_2}(n_1, n_2, \mu_s)$$

$$+ \int_{\mu_s^2}^{Q_2^2} \frac{dk_{2\perp}^2}{k_{2\perp}^2} \Big\{ T_{a_1}(Q_1, \mu_s)\, T_{a_2}(Q_2, k_{2\perp}) \sum_b \tilde{P}_{a_2 b}(n_2, k_{2\perp}) \sum_{a''} \tilde{E}_{b a''}(n_2, k_{2\perp}, \mu_s)\, \tilde{D}^{(sp)}_{a_1 a''}(n_1, n_2, k_{2\perp}) \Big\}$$

$$+ \int_{\mu_s^2}^{Q_1^2} \frac{dk_{1\perp}^2}{k_{1\perp}^2} \Big\{ T_{a_2}(Q_2, \mu_s)\, T_{a_1}(Q_1, k_{1\perp}) \sum_b \tilde{P}_{a_1 b}(n_1, k_{1\perp}) \sum_{a'} \tilde{E}_{b a'}(n_1, k_{1\perp}, \mu_s)\, \tilde{D}^{(sp)}_{a' a_2}(n_1, n_2, k_{1\perp}) \Big\}$$

$$+ \int_{\mu_s^2}^{Q_1^2} \frac{dk_{1\perp}^2}{k_{1\perp}^2} \int_{\mu_s^2}^{Q_2^2} \frac{dk_{2\perp}^2}{k_{2\perp}^2} \Big\{ T_{a_1}(Q_1, k_{1\perp})\, T_{a_2}(Q_2, k_{2\perp}) \sum_{b,c} \tilde{P}_{a_1 b}(n_1, k_{1\perp})\, \tilde{P}_{a_2 c}(n_2, k_{2\perp}) \sum_{a', a''} \tilde{E}_{b a'}(n_1, k_{1\perp}, \mu_s)\, \tilde{E}_{c a''}(n_2, k_{2\perp}, \mu_s)\, \tilde{D}^{(sp)}_{a' a''}(n_1, n_2, \mu_s) \Big\} \Big]$$

Again one ends up with four terms, their importance depends on the scale ordering.

However, due to the internal integration over scale μ_s each of the terms can have perturbative contribution.
In particular first term has no transverse dependence but is perturbative when $\mu_s \sim Q_1 \sim Q_2$

Non-homogeneous part

- Transverse momentum dependence can be also generated when parent parton splits into daughter partons with non-negligible transverse momenta.
- Need to take into account exact kinematics at the splitting vertex.

Diehl, Ostermeier, Schaefer

splitting contribution from gluon splitting

$$f_{a_1a_2}(x_1,x_2,\mathbf{k}_{1\perp},\mathbf{k}_{2\perp},\mathbf{r}) = k_{1\perp}^2 k_{2\perp}^2 \frac{\alpha_s}{4\pi^2}\left[\frac{f_a(x_1+x_2,\kappa_\perp)}{x_1+x_2} T^{ll'}_{a_1a_2}\left(\frac{x_1}{x_1+x_2}\right) + \frac{h_a^\perp(x_1+x_2,\kappa_\perp)}{x_1+x_2} U^{ll'mm'}_{a_1a_2}\left(\frac{x_1}{x_1+x_2}\right) R^{mm'}(M,\kappa_\perp)\right] \frac{(\mathbf{k}_\perp+\frac{1}{2}\mathbf{r})^l(\mathbf{k}_\perp-\frac{1}{2}\mathbf{r})^{l'}}{(\mathbf{k}_\perp+\frac{1}{2}\mathbf{r})^2(\mathbf{k}_\perp-\frac{1}{2}\mathbf{r})^2}$$

$$\mathbf{k}_\perp = \frac{1}{2}(\mathbf{k}_{1\perp}-\mathbf{k}_{2\perp}), \qquad \boldsymbol{\kappa}_\perp = \mathbf{k}_{1\perp}+\mathbf{k}_{2\perp},$$

$f_a(x,\kappa_\perp)$ TMD function

$h^\perp(x,\kappa_\perp)$ Boer-Mulders function

Transverse momentum dependent splitting

Need to consistently combine these terms. Work in progress...

Summary and outlook

- Summary & outlook 1 - initial conditions:
  - Double integrated PDFs need consistent initial conditions for the evolution.
  - Beta functions for single PDF and Dirichlet distributions for double PDF with suitably matched powers and coefficients are good initial conditions. The momentum sum rule and quark number sum rule are satisfied simultaneously.
  - Extending the formalism: expansion in terms of Dirichlet distributions. First numerical tests with gluons. Sum rules provide relations between the powers at small and large x for single and double parton distributions.
  - In principle one can include quarks into the formalism; some additional constraints needed.
  - Is there any deeper physical meaning to the presented algorithm?

Summary and outlook

- Summary & outlook 2 - transverse momentum dependence:
  - First attempt to extend the KMR approach to dPDFs.
  - Homogeneous term quite straightforward, can be implemented numerically. Expression will naturally include correlations through the integrated dPDFs. Additional correlations enter through the regularization cutoffs.
  - In-homogeneous term partially can be treated by the same method.
  - Additional contribution due to perturbative splitting needs to be taken into account. Goes beyond the accuracy of the KMR framework.
  - Need to consistently match different contributions. Perform numerical analysis.

backup

Initial conditions: quarks and gluons

Momentum sum rule with quarks:

$$\sum_{f_1} \tilde{D}_{f_1 f_2}(2, n_2) \;=\; \tilde{D}_{f_2}(n_2) - \tilde{D}_{f_2}(n_2+1)$$

Quark number sum rule:

$$\tilde{D}_{q_i f_2}(1, n_2) - \tilde{D}_{\bar{q}_i f_2}(1, n_2) \;=\; A_{i f_2} \tilde{D}_{f_2}(n_2)$$

$$A_{i f_2} = N_i - \delta_{f_2 q_i} + \delta_{f_2 \bar{q}_i}$$

Ansatz for dPDF with different flavors:

$$D_{f_1 f_2}(x_1, x_2) = N_2\, x_1^{-\tilde{\alpha}^{f_1}}\, x_2^{-\tilde{\alpha}^{f_2}}\, (1 - x_1 - x_2)^{\tilde{\beta}^{f_1 f_2}}$$

Ansatz for sPDF :

$$D_f(x) = N_1\, x^{-\alpha^f}\, (1-x)^{\beta^f}$$

- Can perform the same analysis as before.
- Conditions for powers for dPDFs and sPDFs are exactly the same from both momentum and quark sum rules.
- Can satisfy simultaneously both sum rules:

Small x powers are identical:

$$\tilde{\alpha}^{f_2} = \alpha^{f_2}$$

$$\tilde{\alpha}^{f_1} = \alpha^{f_1}$$

Large x powers:

$$\tilde{\beta}^{f_1 f_2} = \beta^{f_2} + \alpha^{f_1} - 1$$

Symmetry with respect to the parton exchange

$$\tilde{\beta}^{f_1 f_2} = \tilde{\beta}^{f_2 f_1}$$

Implies the correlation of powers in sPDFs:

$$\beta^{f_2} + \alpha^{f_1} = \beta^{f_1} + \alpha^{f_2}$$