

Ivan Vitev

Perturbative and non-perturbative aspects of jet physics in heavy ion collisions at RHIC and LHC

International Symposium on Multi-particle Dynamics 2016,
August-September 2016

Seogwipo, Jeju island, Korea

Plan for the talk

Thanks to the organizer / conveners for the invitation to ISMD2016



- An effective theory for jet propagation in matter
- In-medium splitting functions, DGLAP evolution
- Jet cross sections and jet shapes from SCET_G
- Massive splitting functions and heavy flavor suppression
- New jet substructure observables / soft dropped distributions
- Conclusions

Credit for the work goes to my collaborators: Y.-T. Chien, A. Emerman, Z Kang, R. Lashof-Regs, G. Ovanesyan, Z. Kang, F. Ringer, P. Saad, H. Xing ...

SCET formulation

- Modes in SCET

C. Bauer et al. (2001)

D. Pirol et al. (2004)

Collinear quarks, antiquarks	$\xi_n, \bar{\xi}_n$
Collinear gluons, soft gluons	A_n, A_s

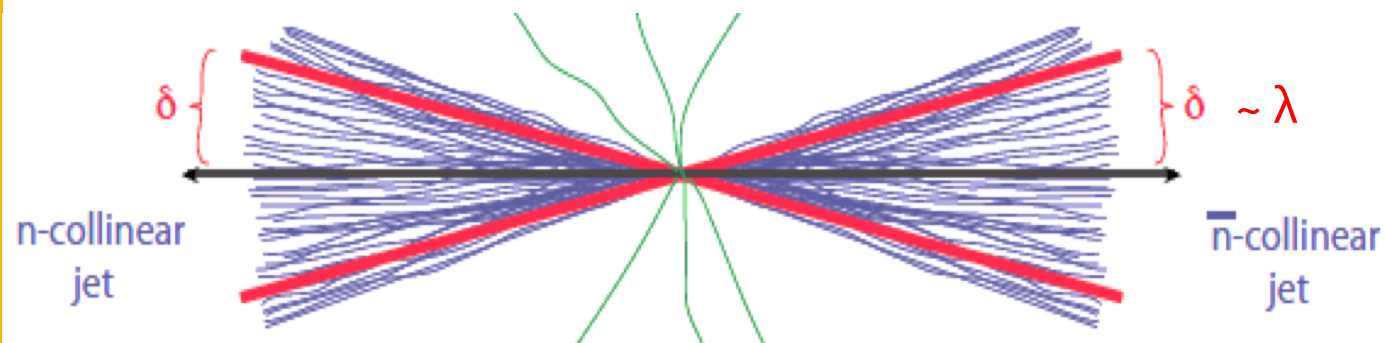
Soft quarks are eliminated through the equations of motion

SCET_{II}

modes	$p^\mu = (+, -, \perp)$	p^2	fields
collinear	$Q(\lambda^2, 1, \lambda)$	$Q^2 \lambda^2$	ξ_n, A_n^μ
soft	$Q(\lambda, \lambda, \lambda)$	$Q^2 \lambda^2$	q_s, A_s^μ

- Other formulations, e.g. SCET_I and ultrasoft particles

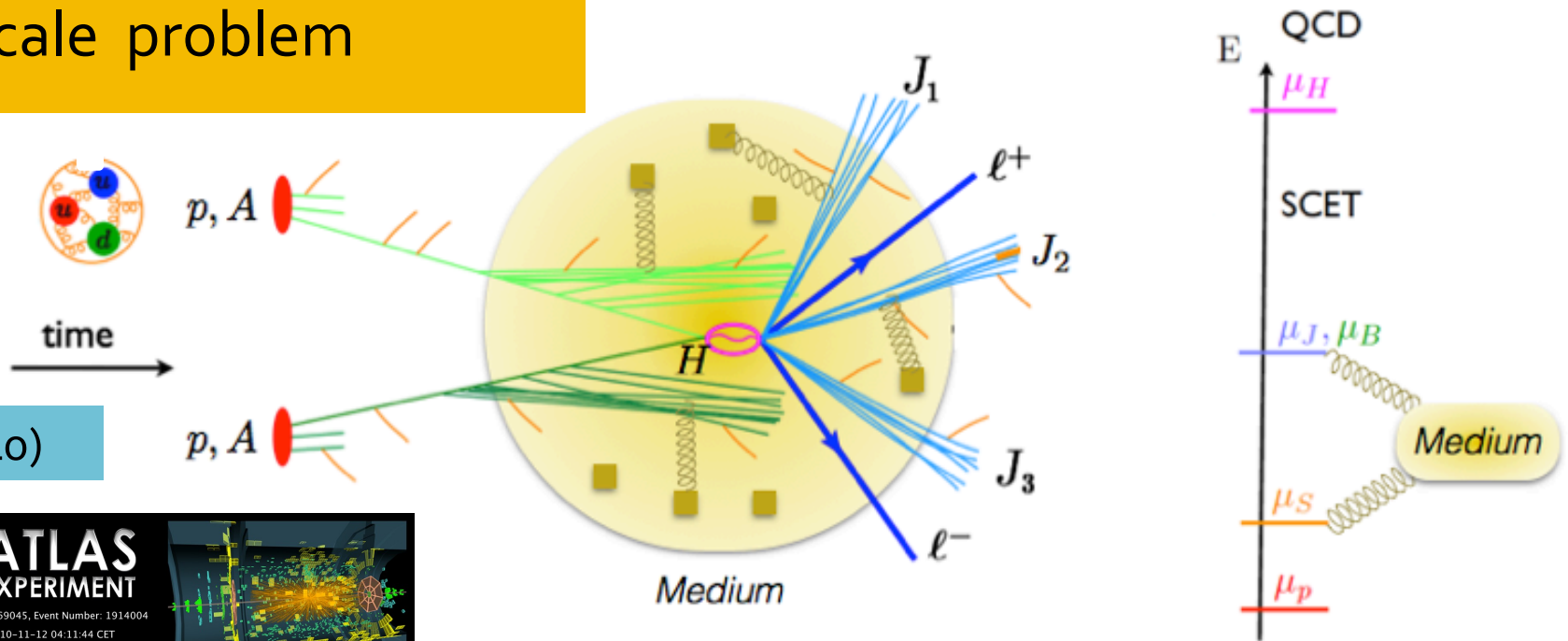
- Especially suited for jet physics. Proofs of factorization and resummation



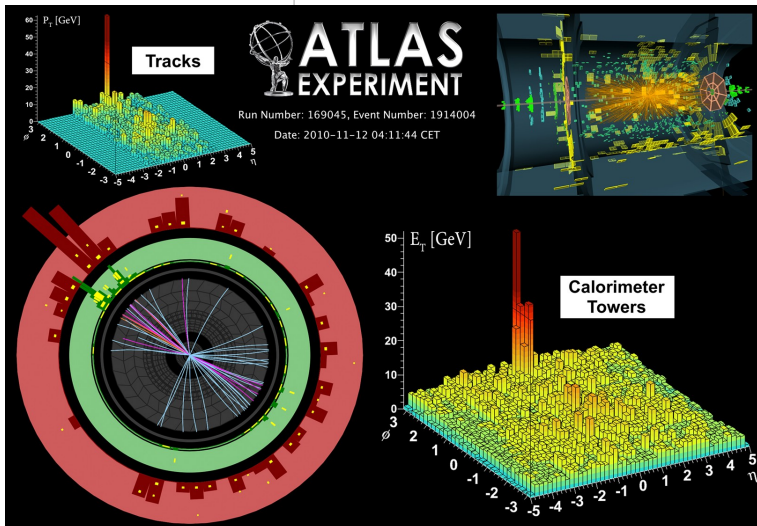
The big picture

- QCD in the medium remains a multiscale problem

Ovanesyan et al. (2011)



Aad et al. (2010)



- Factorization, with modified J (jet), B (beam), S (soft) functions

$$\sigma = \text{Tr}(HS) \otimes \prod_{i=1}^{n_B} B_i \otimes \prod_{j=1}^N J_j + \text{power corrections}$$

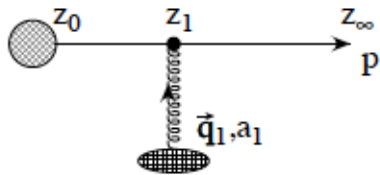
The Glauber gluon Lagrangian

An effective theory of jet propagation in matter
 - couple the collinear and dense QCD sectors

Glauber gluon

$$q = (\lambda^2, \lambda^2, \lambda)Q$$

Effective potential



Forward scattering, t-channel gluon exchanges

$$\mathcal{L}_G(\xi_n, A_n, \eta) = \sum_{p,p',q} e^{-i(p-p'+q)x} \left(\bar{\xi}_{n,p'} \Gamma_{qqA_G}^{\mu,a} \frac{\bar{\eta}}{2} \xi_{n,p} - i \Gamma_{ggA_G}^{\mu\nu\lambda,abc} (A_{n,p'}^c)_\lambda (A_{n,p}^b)_\nu \right) \bar{\eta} \Gamma_s^{\delta,a} \eta \Delta_{\mu\delta}(q)$$

A. Idilbi et al. (2008)

G. Ovanesyan et al. (2011)

$$\Gamma_1^{\mu,a} = igT^a n^\mu \frac{\bar{\eta}}{2},$$

$$\Gamma_2^{\mu,a} = igT^a \frac{\gamma_\perp^\mu \not{p}_\perp + \not{p}'_\perp \gamma_\perp^\mu}{\bar{n} \cdot p} \frac{\bar{\eta}}{2},$$

$$\Gamma_3^{\mu,a} = igT^a v^\mu,$$

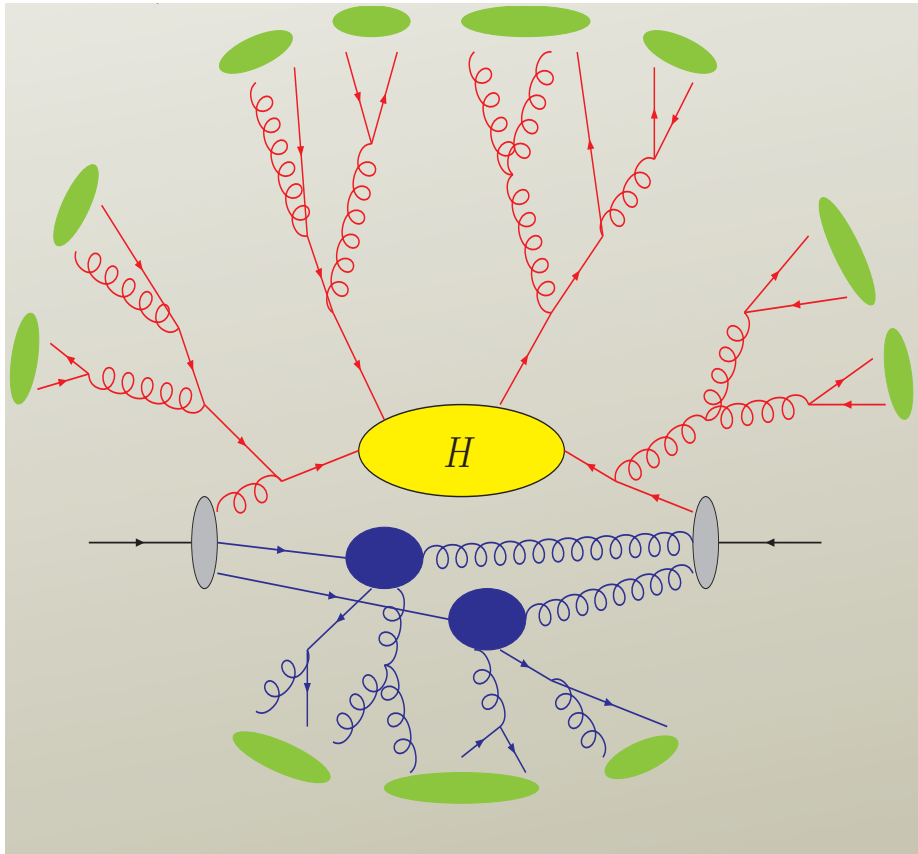
$$\Gamma_4^{\mu,a} = igT^a \gamma^\mu,$$

$$\Sigma_1^{\mu\nu\lambda,abc} = gf^{abc} n^\mu \left[g^{\nu\lambda} \bar{n} \cdot p + \bar{n}^\nu (p'^\lambda - p^\lambda) - \bar{n}^\lambda (p'^\nu - p^\nu) - \frac{1 - \frac{1}{\xi}}{2} (\bar{n}^\lambda p^\nu + \bar{n}^\nu p'^\lambda) \right],$$

G. Ovanesyan et al. (2011)

- Feynman rules for different sources
- and gauges

The splitting kernels



- Splitting functions are related to beam (B) and jet (J) functions in SCET

$$A_{q \rightarrow qg} = \langle J | T \bar{\chi}_n(x_0) e^{iS} | q(\mathbf{p}) g(\mathbf{k}) \rangle$$

$$A_{g \rightarrow q\bar{q}} = \langle J | T \mathcal{B}^{\lambda c}(x_0) e^{iS} | q(\mathbf{p}) \bar{q}(\mathbf{k}) \rangle$$

$$A_{g \rightarrow gg} = \langle J | T \mathcal{B}^{\lambda c}(x_0) e^{iS} | g(\mathbf{p}) g(\mathbf{k}) \rangle$$

$$A_1^{(0)} = \text{Diagram showing a jet function } J \text{ at } x_0 \text{ with an incoming beam } p \text{ and an outgoing gluon } k.$$

$$A_2^{(0)} = \text{Diagram showing a jet function } J \text{ at } x_0 \text{ with an incoming beam } p \text{ and an outgoing quark } k.$$

$$\Gamma_W^{\alpha,a}(k) = g T_r^a \frac{\bar{n}^\alpha}{k^+ + i\epsilon}$$

Gribov et al. (1972)
 G. Altarelli et al. (1977)
 Y. Dokshitzer (1977)

- In the vacuum we have the DGPAL splitting kernels that factorize from the hard scattering cross section and are process independent

In-medium parton splittings and their properties

- Direct sum

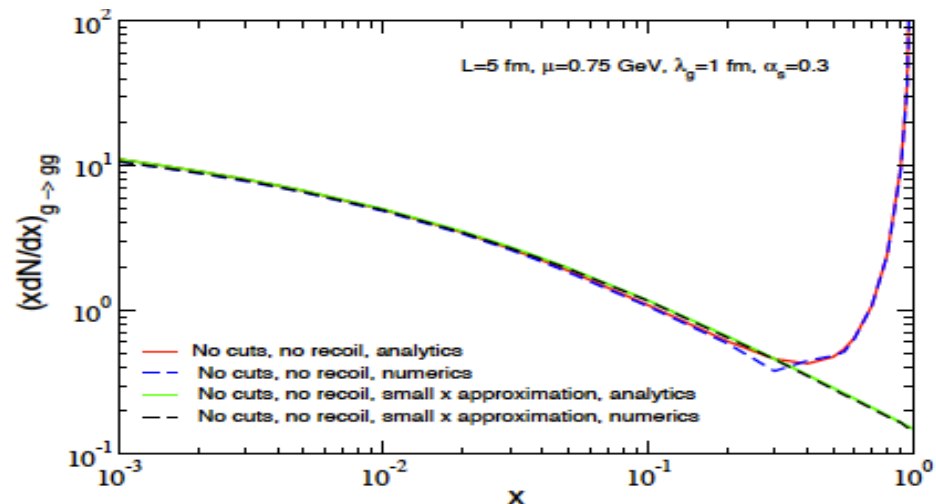
$$\frac{dN(\text{tot.})}{dx d^2k_{\perp}} = \frac{dN(\text{vac.})}{dx d^2k_{\perp}} + \frac{dN(\text{med.})}{dx d^2k_{\perp}}$$

- Derived using SCET_G
- Factorize form the hard part
- Gauge-invariant
- Depend on the properties of the medium

$$\begin{aligned} \left(\frac{dN}{dx d^2k_{\perp}}\right)_{q \rightarrow qg} &= \frac{\alpha_s}{2\pi^2} C_F \frac{1+(1-x)^2}{x} \int \frac{d\Delta z}{\lambda_g(z)} \int d^2\mathbf{q}_{\perp} \frac{1}{\sigma_{el}} \frac{d\sigma_{el}^{\text{medium}}}{d^2\mathbf{q}_{\perp}} \left[-\left(\frac{A_{\perp}}{A_{\perp}^2}\right)^2 + \frac{B_{\perp}}{B_{\perp}^2} \cdot \left(\frac{B_{\perp}}{B_{\perp}^2} - \frac{C_{\perp}}{C_{\perp}^2}\right) \right. \\ &\times (1 - \cos[(\Omega_1 - \Omega_2)\Delta z]) + \frac{C_{\perp}}{C_{\perp}^2} \cdot \left(2\frac{C_{\perp}}{C_{\perp}^2} - \frac{A_{\perp}}{A_{\perp}^2} - \frac{B_{\perp}}{B_{\perp}^2}\right) (1 - \cos[(\Omega_1 - \Omega_3)\Delta z]) \\ &+ \frac{B_{\perp}}{B_{\perp}^2} \cdot \frac{C_{\perp}}{C_{\perp}^2} (1 - \cos[(\Omega_2 - \Omega_3)\Delta z]) + \frac{A_{\perp}}{A_{\perp}^2} \cdot \left(\frac{A_{\perp}}{A_{\perp}^2} - \frac{D_{\perp}}{D_{\perp}^2}\right) \cos[\Omega_4\Delta z] \\ &\left. + \frac{A_{\perp}}{A_{\perp}^2} \cdot \frac{D_{\perp}}{D_{\perp}^2} \cos[\Omega_5\Delta z] + \frac{1}{N_c^2} \frac{B_{\perp}}{B_{\perp}^2} \cdot \left(\frac{A_{\perp}}{A_{\perp}^2} - \frac{B_{\perp}}{B_{\perp}^2}\right) (1 - \cos[(\Omega_1 - \Omega_2)\Delta z]) \right]. \end{aligned}$$

N.B. $x \rightarrow 1-x$ $A, \dots, D, \Omega_1 \dots \Omega_5$ – functions(x, k_{\perp}, q_{\perp})

Example why traditional energy loss interpretation is not possible in a unified parton shower picture



Evolution of the fragmentation functions

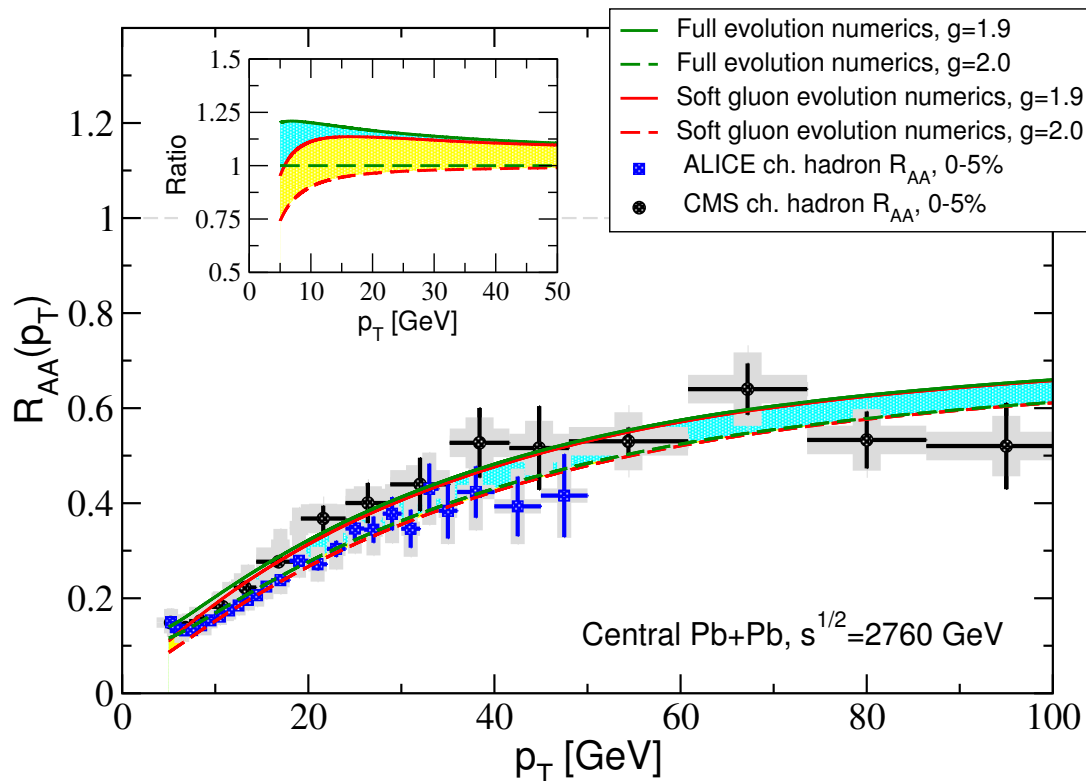
- Yield LLA or MLLA (LL')

Z. Kang et al. (2014)

$$\frac{dD_q(z, Q)}{d \ln Q} = \frac{\alpha_s(Q^2)}{\pi} \int_z^1 \frac{dz'}{z'} \left\{ P_{q \rightarrow qg}(z', Q) D_q\left(\frac{z}{z'}, Q\right) + P_{q \rightarrow gq}(z', Q) D_g\left(\frac{z}{z'}, Q\right) \right\},$$

$$\frac{dD_{\bar{q}}(z, Q)}{d \ln Q} = \frac{\alpha_s(Q^2)}{\pi} \int_z^1 \frac{dz'}{z'} \left\{ P_{q \rightarrow qg}(z', Q) D_{\bar{q}}\left(\frac{z}{z'}, Q\right) + P_{q \rightarrow gq}(z', Q) D_g\left(\frac{z}{z'}, Q\right) \right\},$$

$$\frac{dD_g(z, Q)}{d \ln Q} = \frac{\alpha_s(Q^2)}{\pi} \int_z^1 \frac{dz'}{z'} \left\{ P_{g \rightarrow gg}(z', Q) D_g\left(\frac{z}{z'}, Q\right) + P_{g \rightarrow q\bar{q}}(z', Q) \left(D_q\left(\frac{z}{z'}, Q\right) + \bar{q} \text{ term} \right) \right\}.$$



In the medium: effective thermal masses, finite α_s

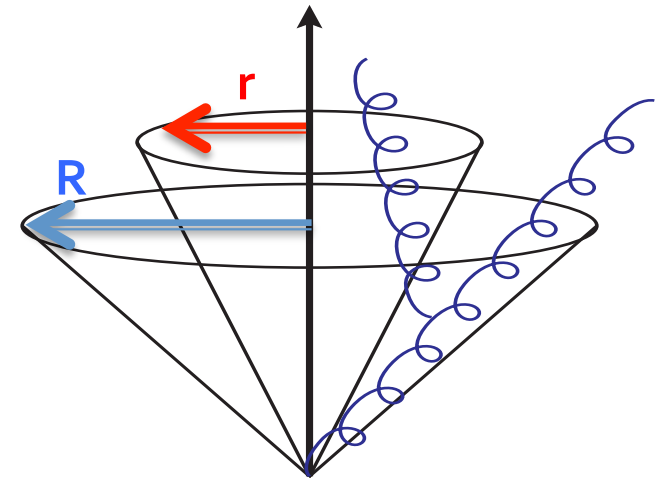
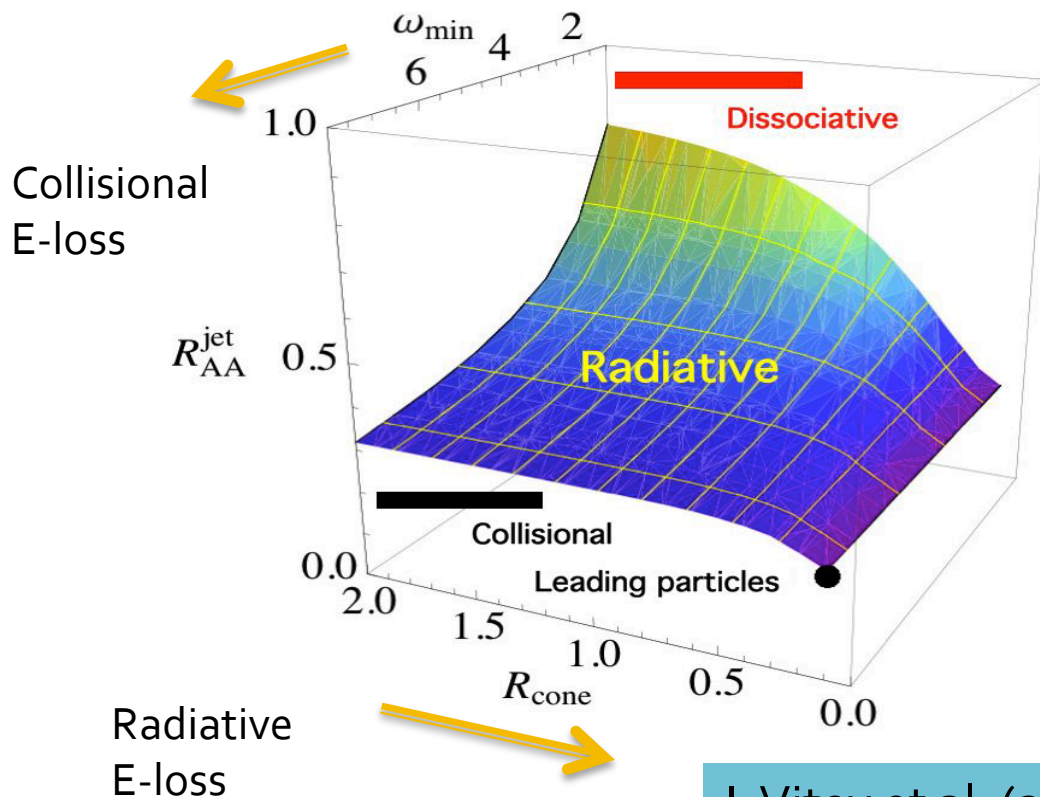
Implement medium –induced splittings as corrections to vacuum evolution

Demonstrated connection to E-loss

Applications of SCET_G to jet shapes and jet cross sections

- Jet cross sections reflect the total amount of energy retained in the jet cone

- Jet shapes reflect the energy density inside the jet and the structure of the parton shower



$$\Psi_{\text{int}}(r; R) = \frac{\sum_i (E_T)_i \Theta(r - (R_{\text{jet}})_i)}{\sum_i (E_T)_i \Theta(R - (R_{\text{jet}})_i)},$$

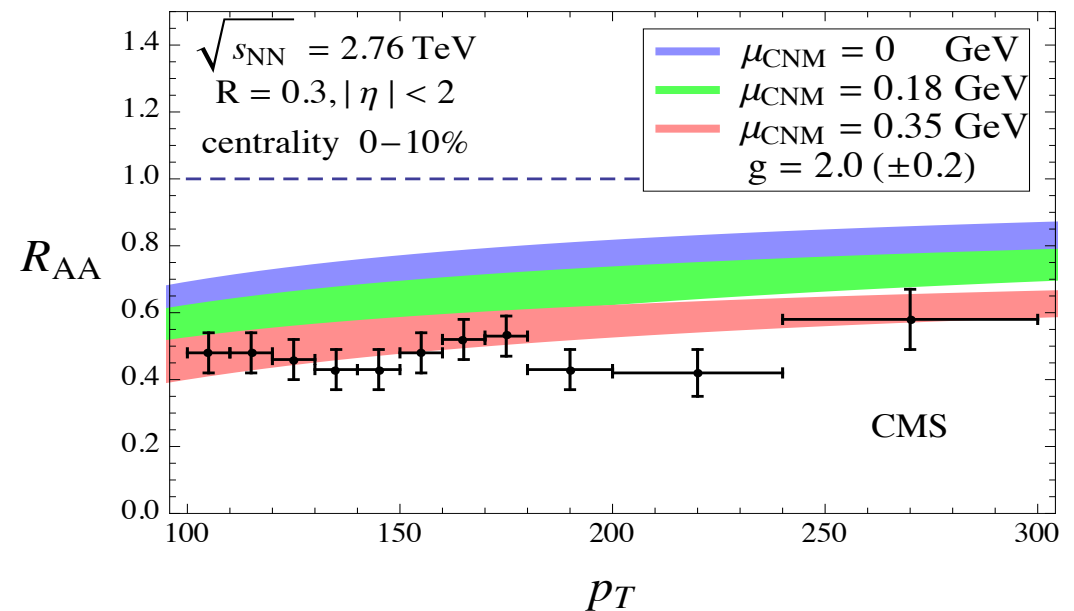
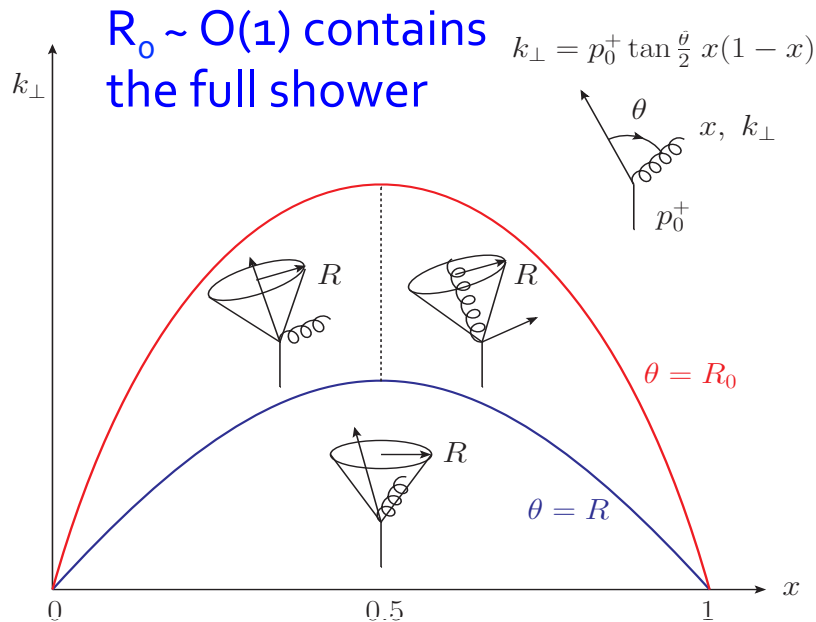
$$\psi(r; R) = \frac{d\Psi_{\text{int}}(r; R)}{dr}.$$

I. Vitev et al. (2008)

Generalizing the concept of energy loss to jets

Y.-T. Chien et al. (2015)

- The jet definition allows to generalize the concept of energy loss

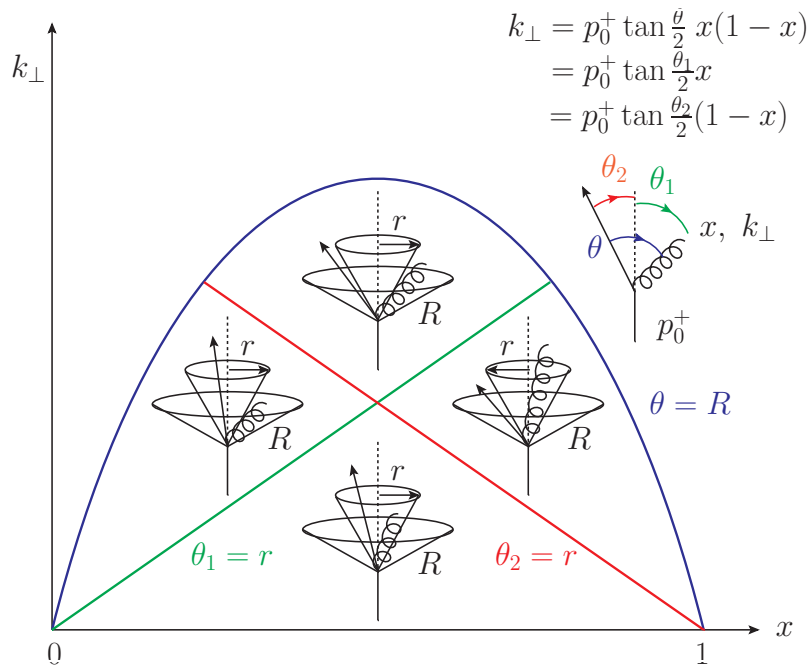


$$\epsilon_q = \frac{2}{\omega} \left[\int_0^{\frac{1}{2}} dx k^0 + \int_{\frac{1}{2}}^1 dx (p^0 - k^0) \right] \int_{\omega x(1-x) \tan \frac{R}{2}}^{\omega x(1-x) \tan \frac{R_0}{2}} dk_{\perp} \frac{1}{2} \left[\mathcal{P}_{q \rightarrow qq}^{\text{med}}(x, k_{\perp}) + \mathcal{P}_{q \rightarrow gq}^{\text{med}}(x, k_{\perp}) \right]$$

$$\epsilon_g = \frac{2}{\omega} \left[\int_0^{\frac{1}{2}} dx k^0 + \int_{\frac{1}{2}}^1 dx (p^0 - k^0) \right] \int_{\omega x(1-x) \tan \frac{R}{2}}^{\omega x(1-x) \tan \frac{R_0}{2}} dk_{\perp} \frac{1}{2} \left[\mathcal{P}_{g \rightarrow gg}^{\text{med}}(x, k_{\perp}) + \sum_{q, \bar{q}} \mathcal{P}_{g \rightarrow q\bar{q}}^{\text{med}}(x, k_{\perp}) \right]$$

Fractional energy loss outside of the jet beyond the soft gluon approximation

Medium-modified jet shapes at NLL



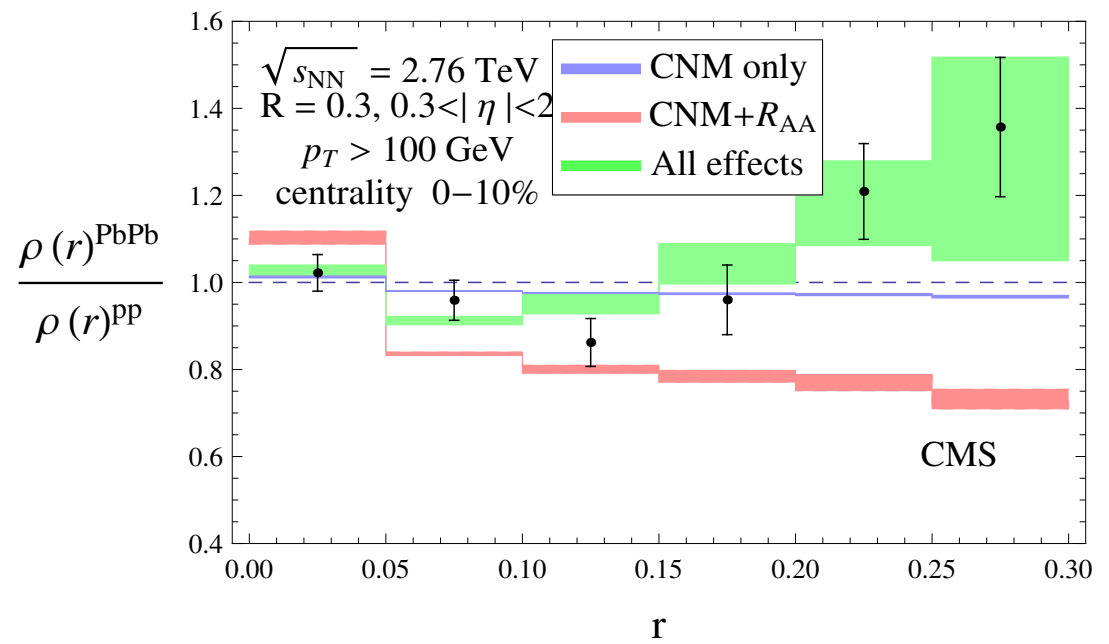
$$E_r(x, k_{\perp}) = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4$$

Measurement operator – tells us how the above configurations contribute energy to J (jet function)

- One can evaluate the jet energy functions from the splitting functions

$$J_{\omega, E_r}^i(\mu) = \sum_{j,k} \int_{PS} dx dk_{\perp} \mathcal{P}_{i \rightarrow jk}(x, k_{\perp}) E_r(x, k_{\perp})$$

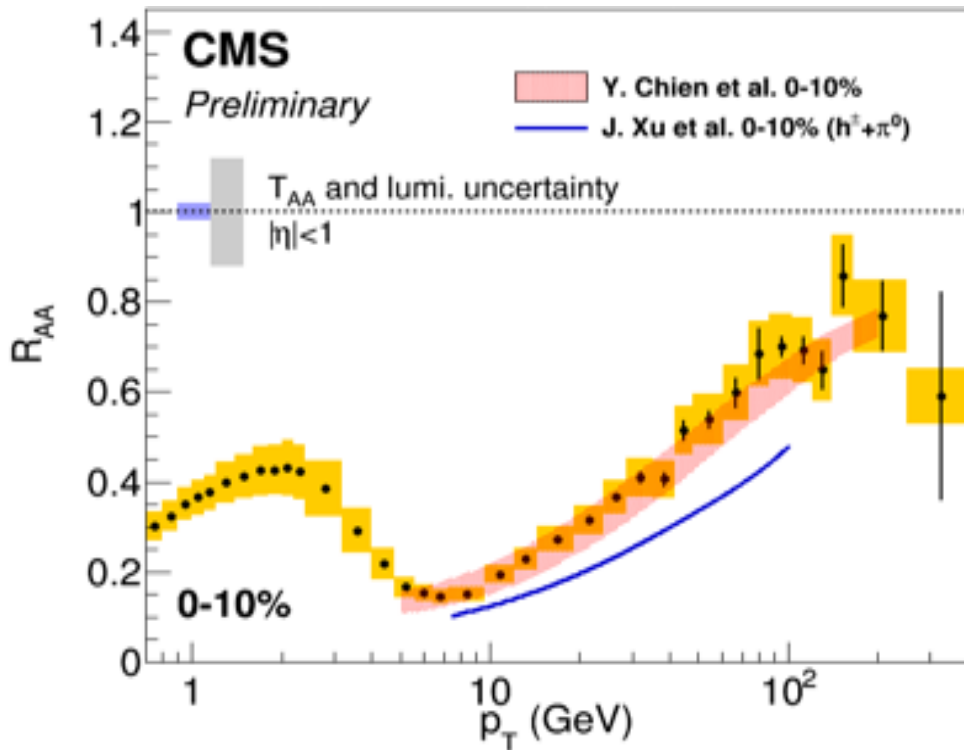
$$J_{\omega, E_r}(\mu) = J_{\omega, E_r}^{vac}(\mu) + J_{\omega, E_r}^{med}(\mu).$$



- First quantitative pQCD/SCET description of jet shapes in HI

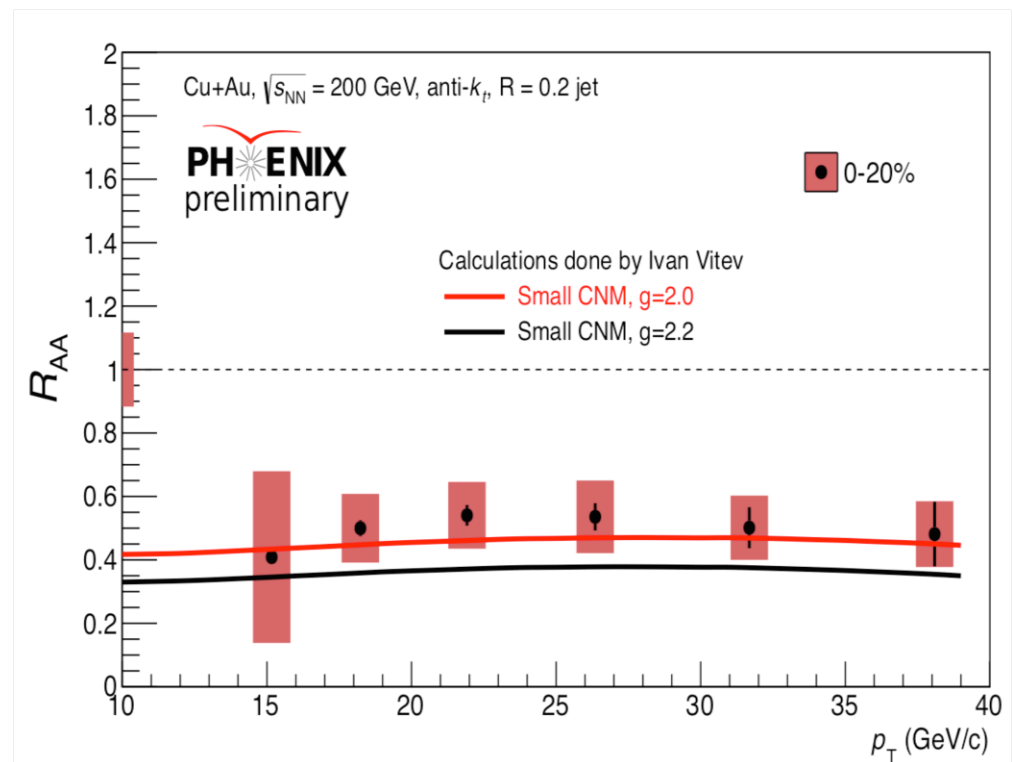
Predictions for HIC beyond E-loss

- Inclusive charged hadron production (and also π^0) at 5.02 TeV in Pb+Pb



Y.-T. Chien et al. (2015)

- Jet production in Cu+Au collisions at 200 GeV. Also γ -jet at the LHC



Y.-T. Chien et al. (2015) (different paper)

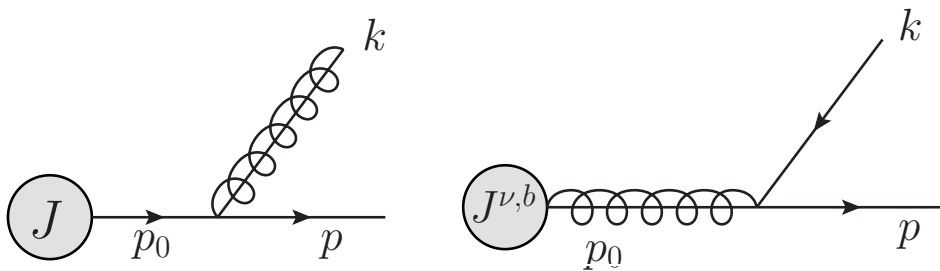
Heavy quarks in the vacuum

3 splitting functions (g to gg is the same)

$$\left(\frac{dN}{dx d^2k_{\perp}}\right)_{Q \rightarrow Qg} = C_F \frac{\alpha_s}{\pi^2} \frac{1}{k_{\perp}^2 + x^2 m^2} \left[\frac{1-x+x^2/2}{x} - \frac{x(1-x)m^2}{k_{\perp}^2 + x^2 m^2} \right]$$

$$\left(\frac{dN}{dx d^2k_{\perp}}\right)_{g \rightarrow Q\bar{Q}} = T_R \frac{\alpha_s}{2\pi^2} \frac{1}{k_{\perp}^2 + m^2} \left[x^2 + (1-x)^2 + \frac{2x(1-x)m^2}{k_{\perp}^2 + m^2} \right]$$

The process is not written Q to gQ but it should have been since x goes to $1-x$

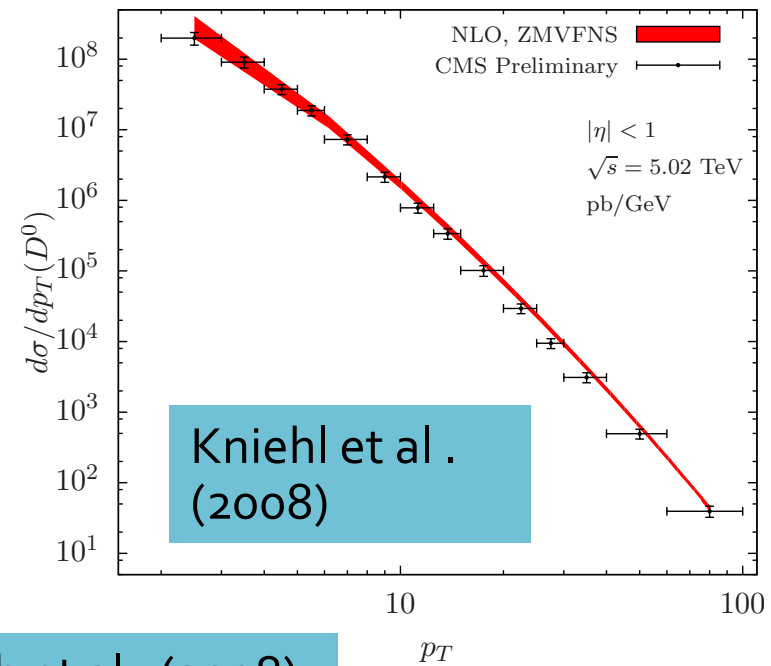
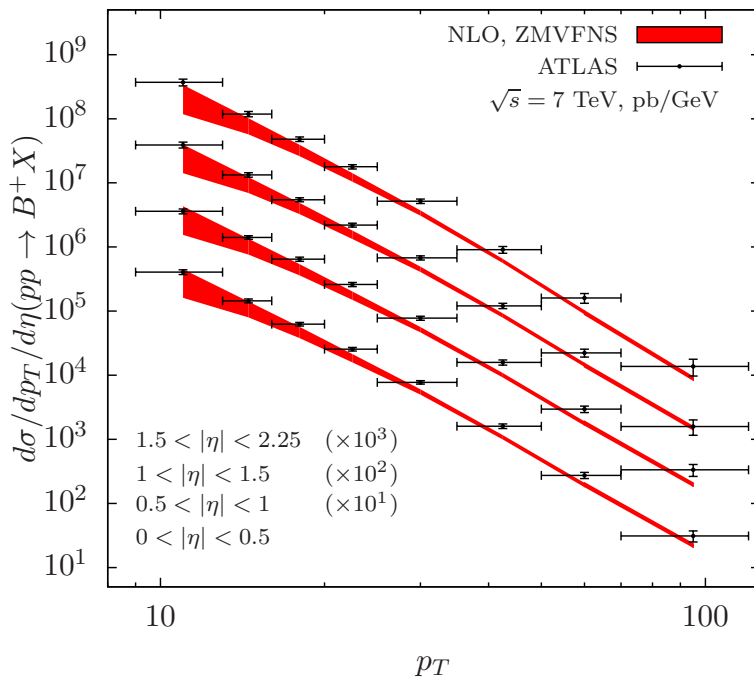


F. Ringer et al. (2016)

- You see the dead cone effects
Dokshitzer et al. (2001)
- You also see that it depends on the process – it not simply $x^2 m^2$ everywhere: $x^2 m^2, (1-x)^2 m^2, m^2$

The medium-induced splitting kernels are now derived (1st order in opacity). More complicated than the vacuum ones. Have been numerically evaluated

ZMVFS open heavy flavor at NLO



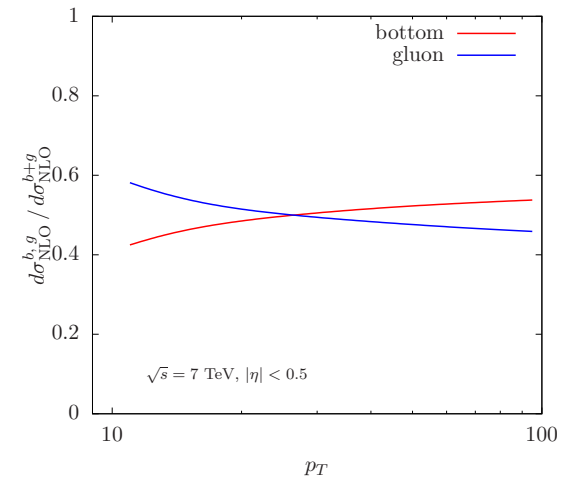
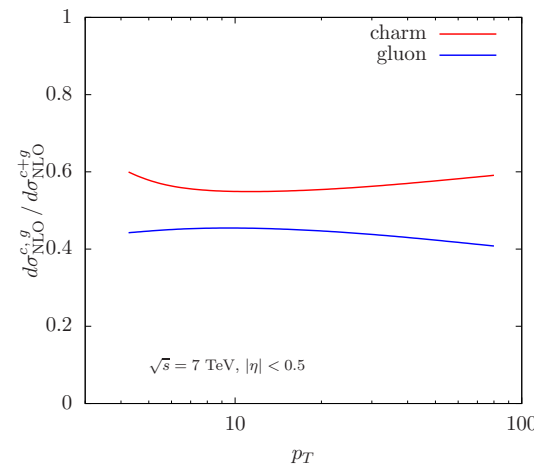
Kniehl et al. (2008)

Kneesch et al. (2008)

When $p_T > m_c, m_b$

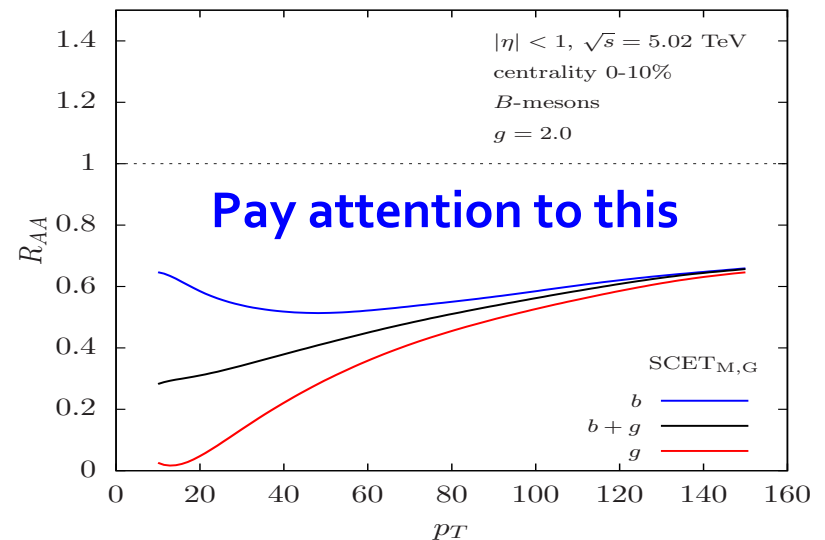
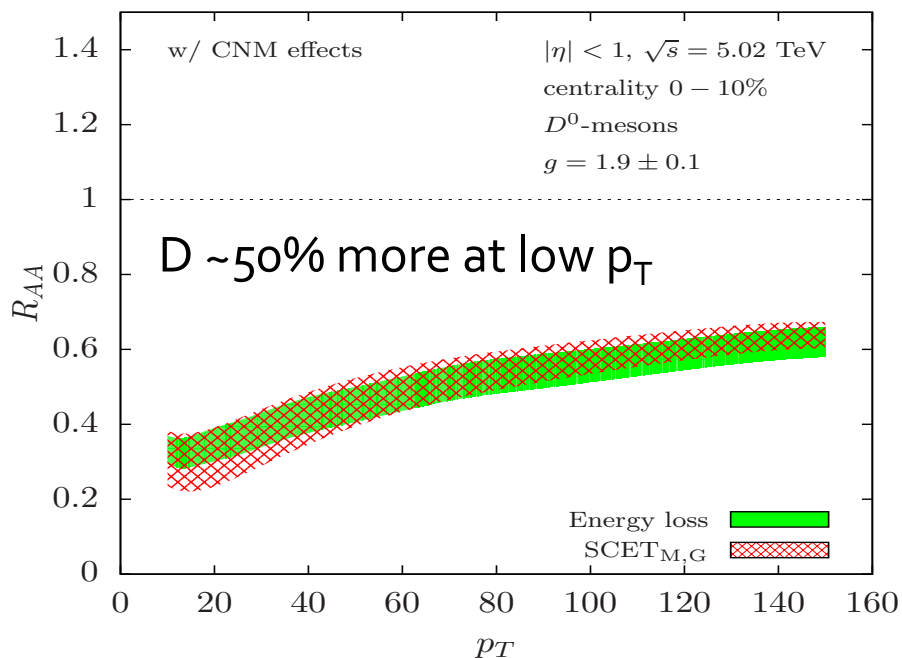
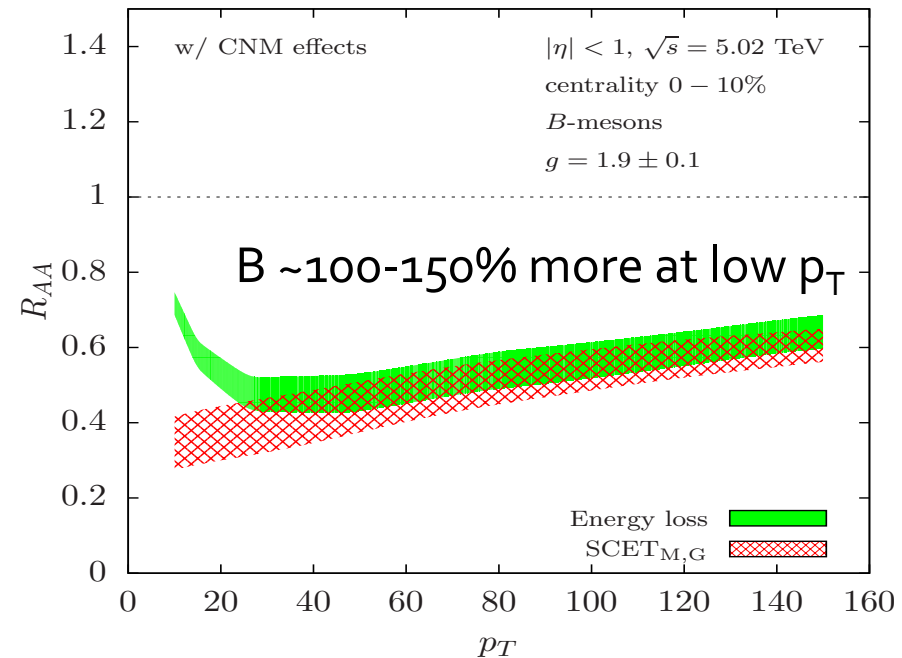
- Perform and NLO calculation
- A very large contribution of gluon FF to heavy flavor

F. Ringer et al. (2016)



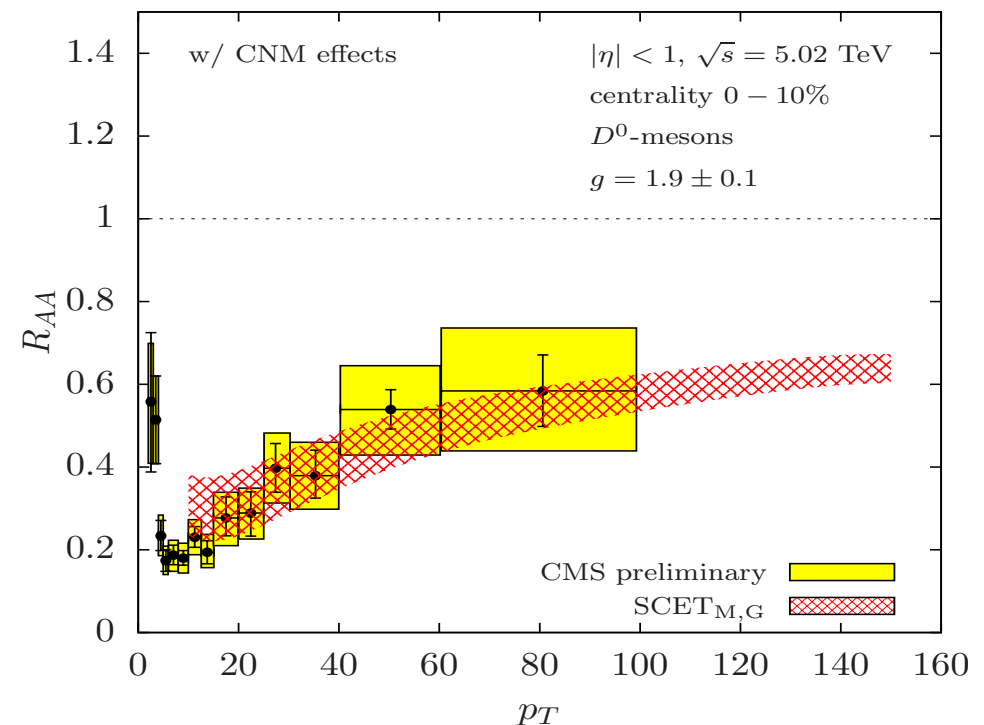
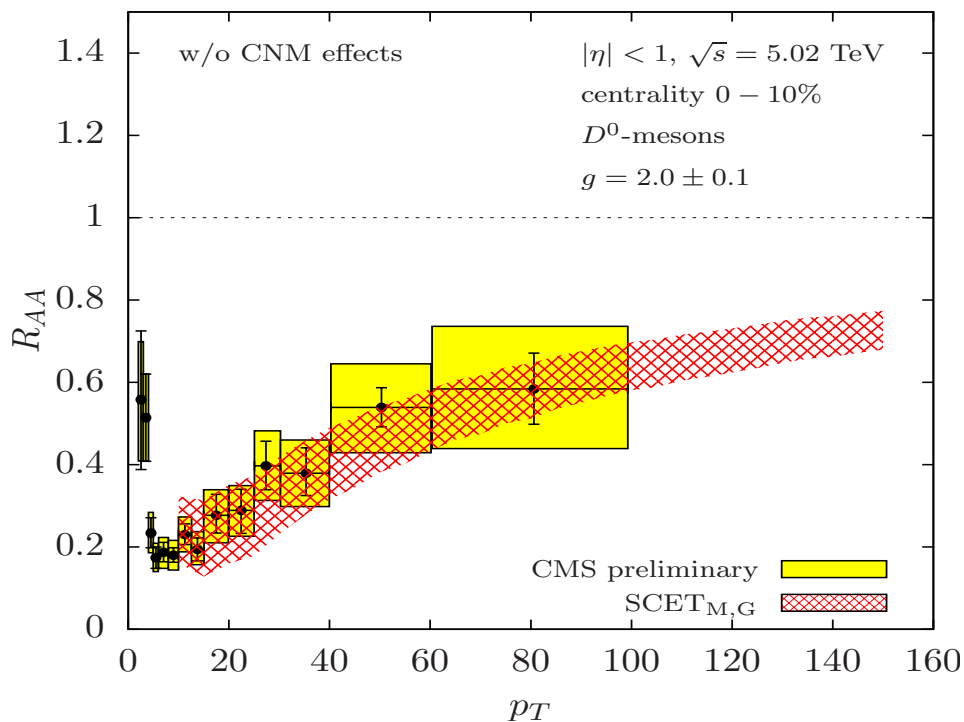
Comparison to the traditional energy loss

- Traditional energy loss approach – charm, bottom quark energy loss
- Full massive splitting function approach. Expansion of DGLAP to first fixed order. Most important is the **gluon contribution and “quenching”**



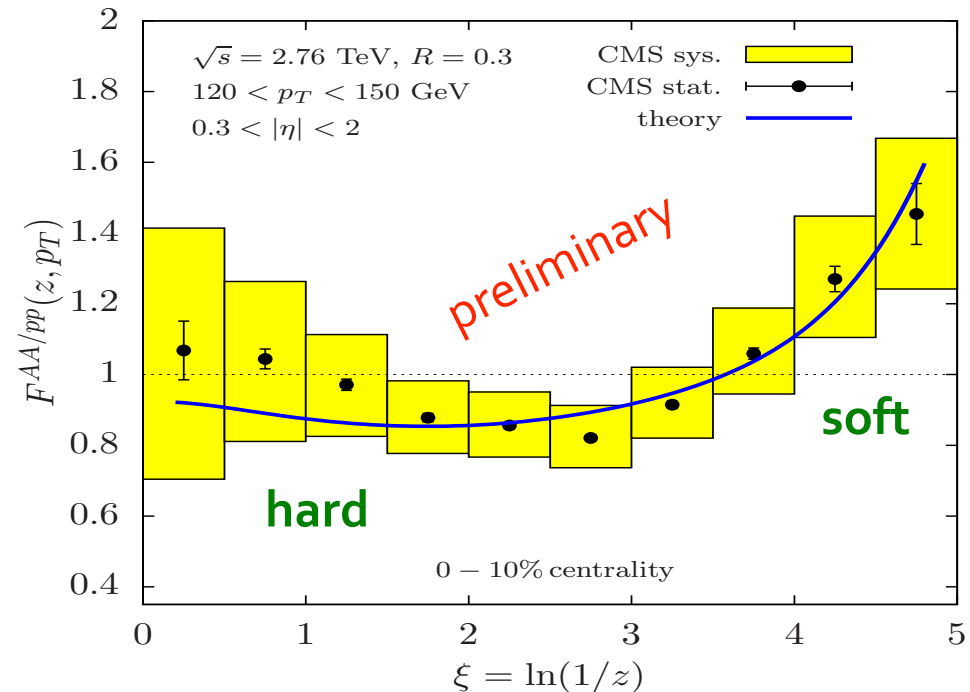
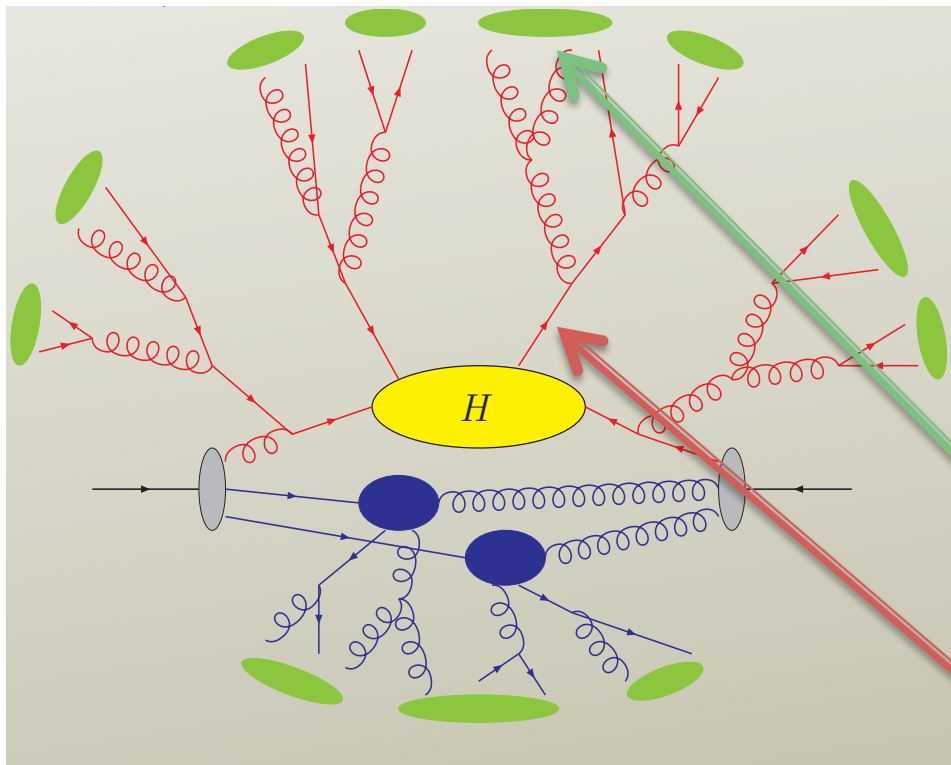
Comparison to Pb+Pb at 5.02 TeV

- We stop at 5 – 7 GeV. It is still important to investigate collisional energy loss, heavy flavor dissociation, for low p_T A. Andronic et al. (2015)
- Nonetheless most of the discrepancy is gone in radiative (medium induced splitting calculations alone) at intermediate p_T and above
- Concern (or shall I say food for thought) about low p_T D,B meson production



Probing the hardest splitting in jets in heavy ion collisions

Jet substructure modification in HIC well established: jet shapes, jet fragmentation functions



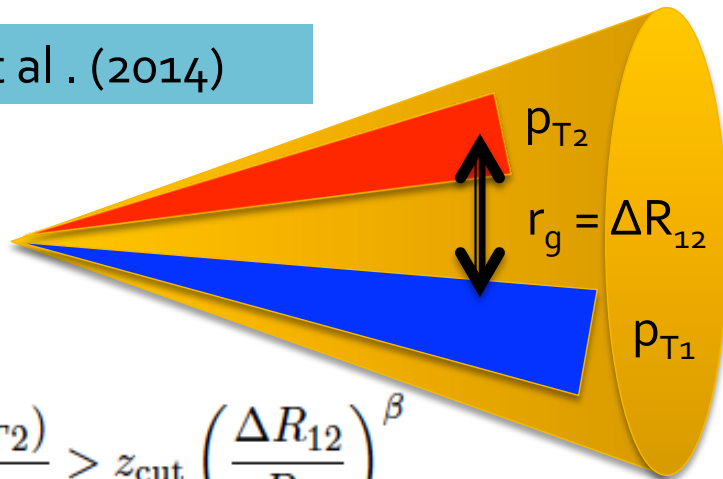
Y. T Chien et al. in progress

Is substructure modification set by late time soft gluon emission?
Or is it manifest in the hard early time splittings?

Many observables to access jet substructure have emerged in SCET

Groomed jet distribution using “soft drop”

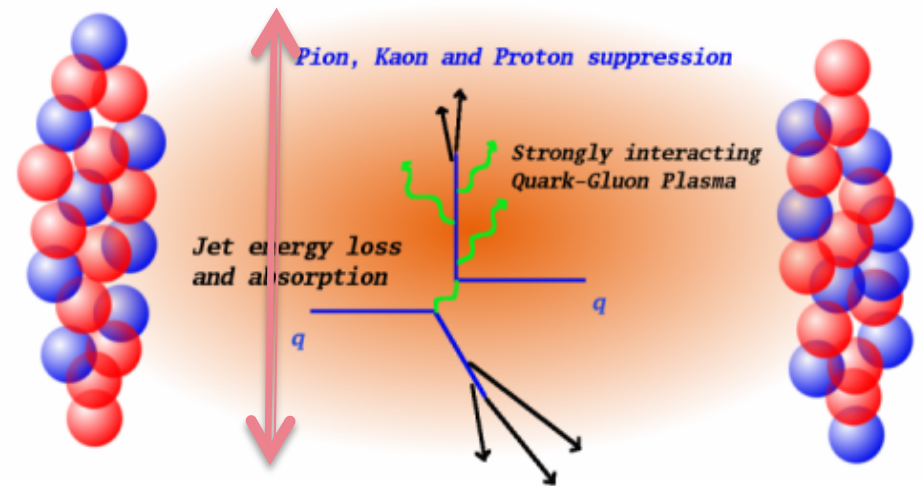
A. Larkoski et al. (2014)



$$z_g = \frac{\min(p_{T1}, p_{T2})}{p_{T1} + p_{T2}} > z_{\text{cut}} \left(\frac{\Delta R_{12}}{R_0} \right)^\beta$$

The great utility of these new distributions: probe the early time dynamics / splitting

$$\tau_{\text{br}}[\text{fm}] = \frac{0.197 \text{ GeV fm}}{z_g(1 - z_g) \omega[\text{GeV}] \tan^2(r_g/2)}$$



QGP size ~ 10fm

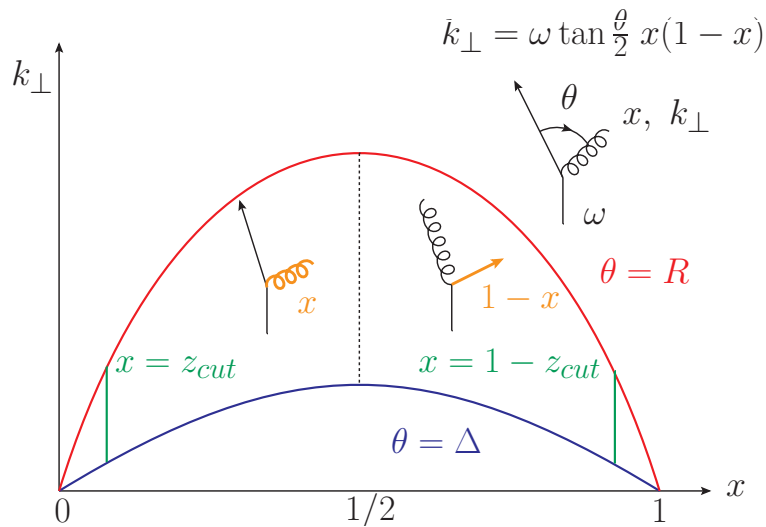
Typical situation: $E=200 \text{ GeV}$, $r_g = 0.1$

Branching time $< 2 \text{ fm}$ for z_g studied

Y. T. Chien et al. (2016)

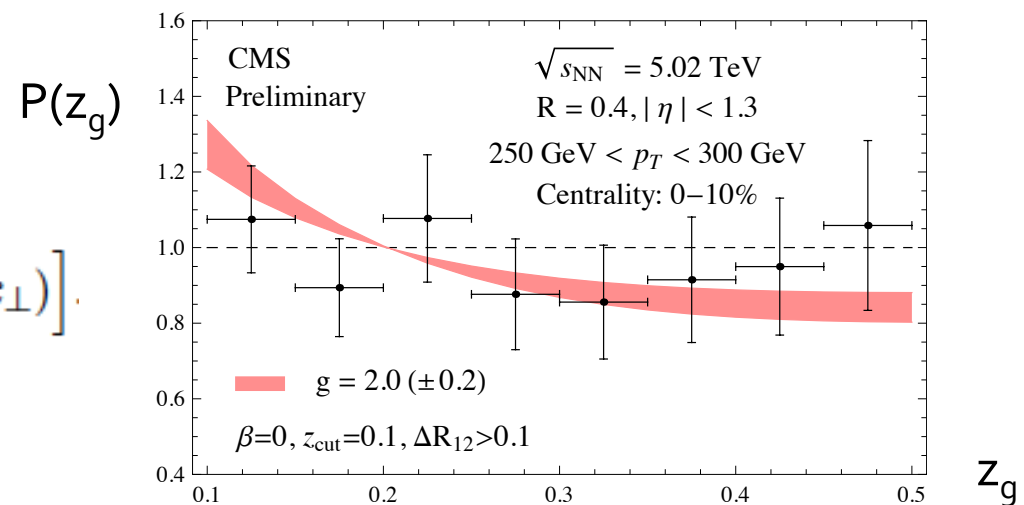
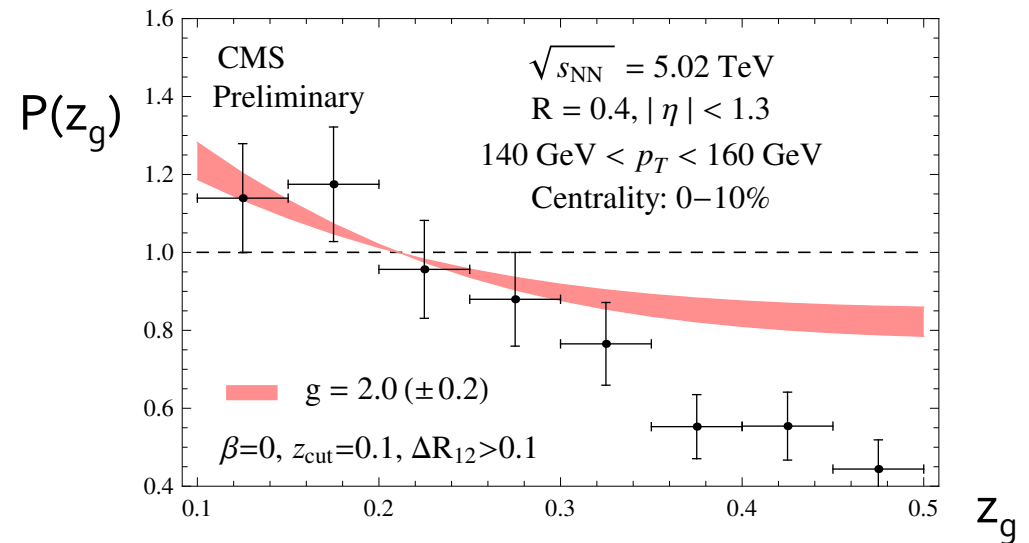
Accessing the hardest branching in HIC – longitudinal modification

Calculating the soft dropped distribution with $\beta=0$

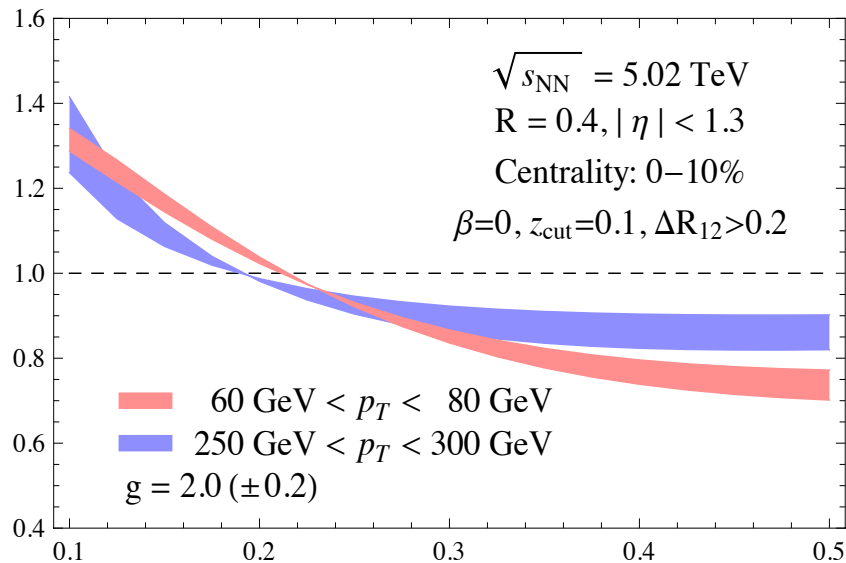


$$P_i(z_g) = \frac{\int_{k_{\Delta}}^{k_R} dk_{\perp} \bar{P}_i(z_g, k_{\perp})}{\int_{z_{cut}}^{1/2} dx \int_{k_{\Delta}}^{k_R} dk_{\perp} \bar{P}_i(x, k_{\perp})}$$

$$\bar{P}_i(x, k_{\perp}) = \sum_{j,l} \left[P_{i \rightarrow j,l}(x, k_{\perp}) + P_{i \rightarrow j,l}(1-x, k_{\perp}) \right].$$



Modification of the angular distribution of hardest branchings



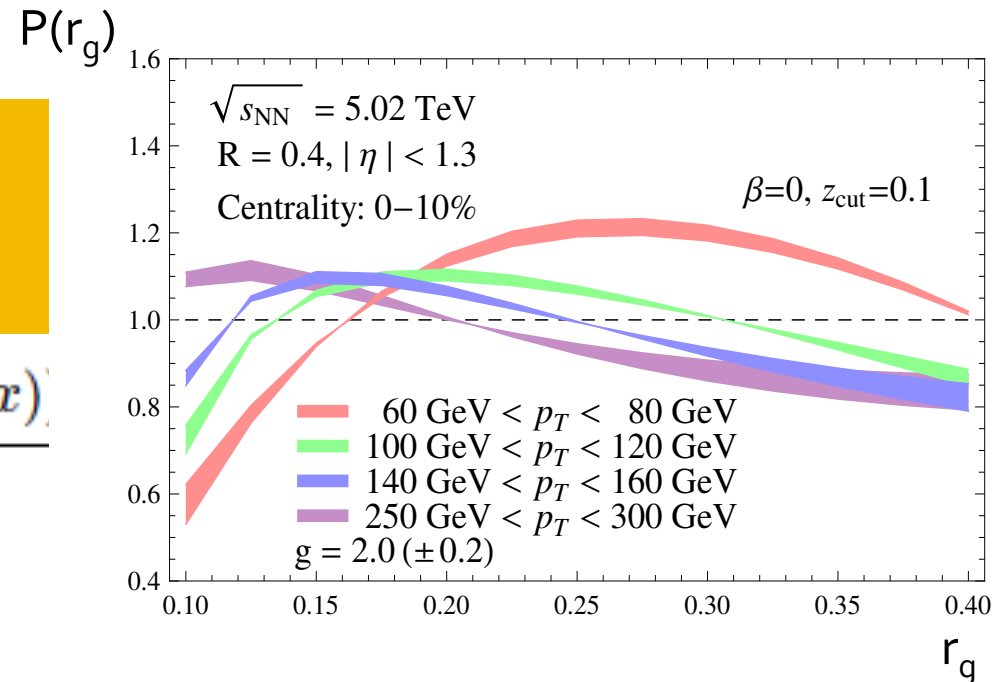
Flexibility in selecting angular separation r_g

Found that intermediate values $r_g = 0.2$ give the strongest p_T dependence. Though not nearly as strong as preliminary data

New observable proposed – measures the typical splitting angle modification in HIC

$$p_i(r_g) = \frac{\int_{z_{cut}}^{1/2} dx p_T x(1-x) \bar{\mathcal{P}}_i(x, k_{\perp}(r_g, x))}{\int_{z_{cut}}^{1/2} dx \int_{k_{\Delta}}^{k_R} dk_{\perp} \bar{\mathcal{P}}_i(x, k_{\perp})}$$

Y.-T. Chien et al. (2016)



Conclusions

- An effective theory of jet propagation in matter SCET_G was constructed (collinear sector). All medium-induced parton splittings derived, factorization and gauge invariance proven
- Unified treatment of parton showers, corrections to DGLAP evolution. The connection to the traditional energy loss established. Excellent agreement between theory and data for inclusive hadron suppression, predictions for the 5.02 TeV run
- Calculations of jet cross sections and jet shapes (substructure) are now available beyond the energy loss approach. Comparable description of inclusive jet suppression to the energy loss approach. Much improved description of jet shape modification
- Derived all massive in-medium splitting kernels beyond energy loss. In phenomenology – need for improved HF production. Large gluon contribution to HF corroborated by b-jet, jet HH and even inclusive hadron production. Much improved description in intermediate p_T
- First application to some of the new substructure observables – groomed soft dropped distribution. The hardest early time splitting is significantly modified suggesting the parton shower modification happens early on. New observables are proposed to test the angular structure of such branchings

2017 Jets and heavy flavor workshop

- Second in a series of workshops to bring the NP and HEP communities working on jets and heavy flavor, with emphasis on QCD and SCET

Santa Fe Jets and Heavy Flavor Workshop

February 13-15, 2017

Workshop topics:

- Jets and jet substructure in hadronic and nuclear collisions
- Heavy flavor production in p+p, p+A and A+A
- Perturbative QCD and SCET
- New theoretical developments
- Recent experimental results from RHIC and LHC

Contact: sfjet17@lanl.gov

Organizers:

Cesar da Silva
Zhongbo Kang
Christopher Lee
Michael McCumber
Duff Neil
Felix Ringer
Ivan Vitev (Chair)

Sponsors:

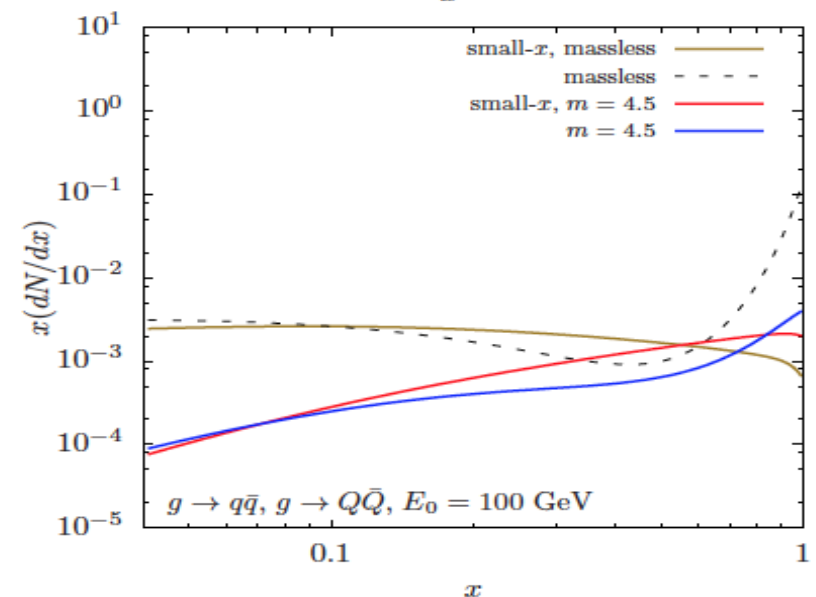
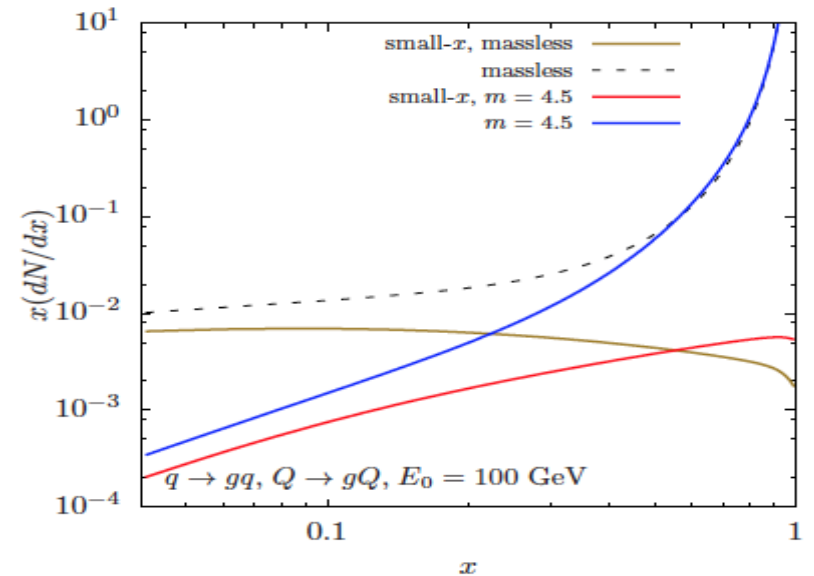
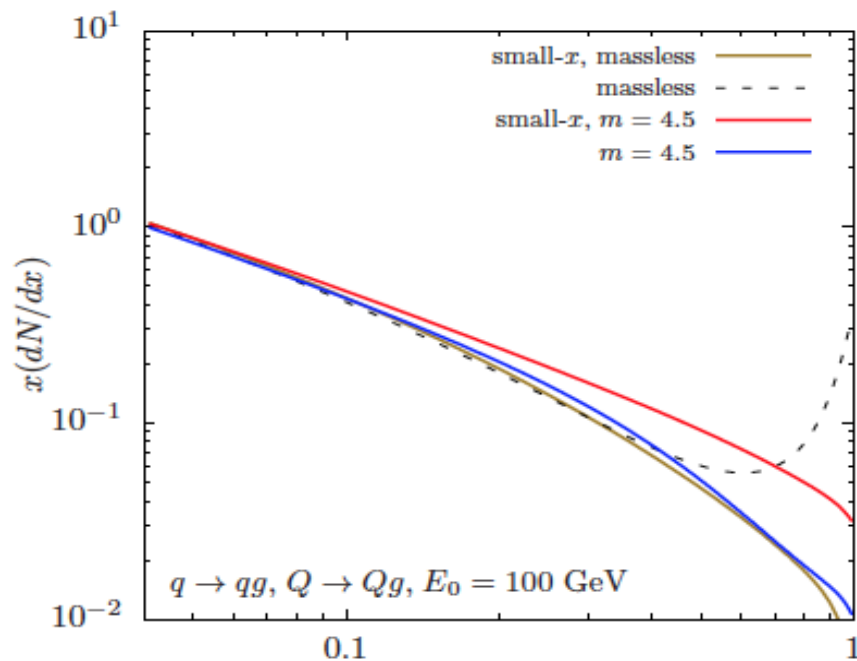
DOE Office of Science
DOE Early Career Program
Los Alamos National Laboratory



Results for the massive in-medium splitting intensities

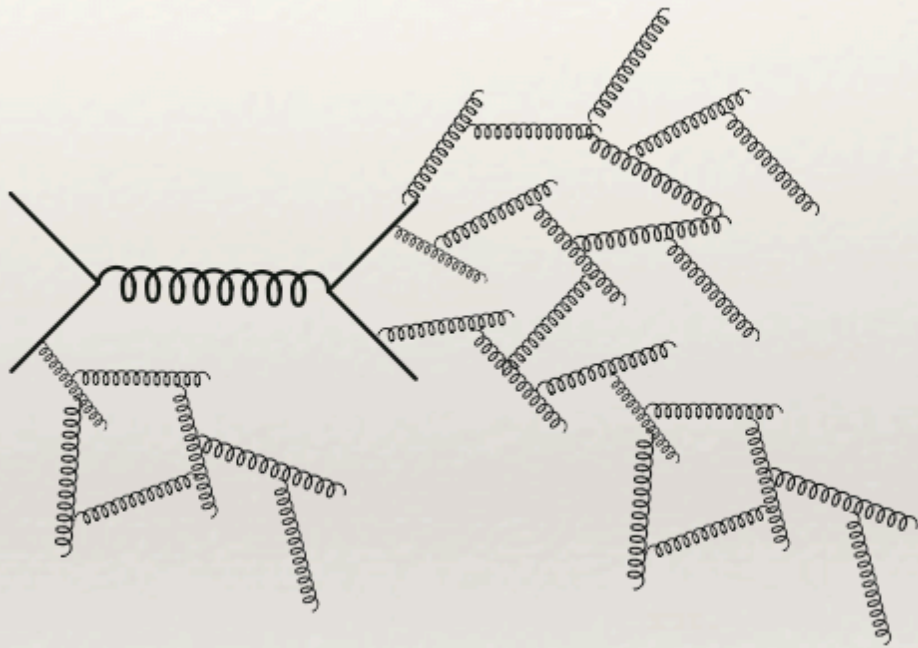
The massive in-medium splitting functions differ considerably from the massless ones

The differences persist even for large energies ($E=100$ GeV)



Logs, legs and loops

Final-state parton shower



Initial-state radiation

- In the description of high energy processes significant effort has been devoted to understand the logs, legs and loops

- Log - ratios of mass and energy scales, phase space, cuts. Goal is to resum
- Legs – the formation of parton shower, branchings, evolution
- Loops – virtual corrections. Goal is to include, find automated way to do some of the loops

- They are connected, one of the goals is to see if some of the technology can be ported to heavy ion collisions

Heavy quarks in the medium

Kinematic variables

$$A_{\perp} = k_{\perp}, \quad B_{\perp} = k_{\perp} + xq_{\perp}, \quad C_{\perp} = k_{\perp} - (1-x)q_{\perp}, \quad D_{\perp} = k_{\perp} - q_{\perp}.$$

$$\frac{dN}{dx} \sim \left| \begin{array}{c} \text{Diagram 1} \\ + \\ \text{Diagram 2} \\ + \\ \text{Diagram 3} \end{array} \right|^2 + 2\text{Re} \left[\begin{array}{c} \text{Diagram 4} \\ + \\ \text{Diagram 5} \\ + \\ \text{Diagram 6} \\ + \\ \text{Diagram 7} \end{array} \right] \times \text{Diagram 8}$$

$$\Omega_1 - \Omega_2 = \frac{B_{\perp}^2 + \nu^2}{p_0^+ x(1-x)}, \quad \Omega_1 - \Omega_3 = \frac{C_{\perp}^2 + \nu^2}{p_0^+ x(1-x)}, \quad \Omega_4 = \frac{A_{\perp}^2 + \nu^2}{p_0^+ x(1-x)},$$

$$\nu = m \quad (g \rightarrow Q\bar{Q}),$$

$$\nu = xm \quad (Q \rightarrow Qg),$$

$$\nu = (1-x)m \quad (Q \rightarrow gQ),$$

F. Ringer et al. (2016)

$$\begin{aligned} \left(\frac{dN^{\text{med}}}{dx d^2k_{\perp}} \right)_{Q \rightarrow Qg} &= \frac{\alpha_s}{2\pi^2} C_F \int \frac{d\Delta z}{\lambda_g(z)} \int d^2q_{\perp} \frac{1}{\sigma_{el}} \frac{d\sigma_{el}^{\text{med}}}{d^2q_{\perp}} \left\{ \left(\frac{1 + (1-x)^2}{x} \right) \left[\frac{B_{\perp}}{B_{\perp}^2 + \nu^2} \right. \right. \\ &\times \left(\frac{B_{\perp}}{B_{\perp}^2 + \nu^2} - \frac{C_{\perp}}{C_{\perp}^2 + \nu^2} \right) (1 - \cos[(\Omega_1 - \Omega_2)\Delta z]) + \frac{C_{\perp}}{C_{\perp}^2 + \nu^2} \cdot \left(2 \frac{C_{\perp}}{C_{\perp}^2 + \nu^2} - \frac{A_{\perp}}{A_{\perp}^2 + \nu^2} \right. \\ &- \left. \left. \frac{B_{\perp}}{B_{\perp}^2 + \nu^2} \right) (1 - \cos[(\Omega_1 - \Omega_3)\Delta z]) + \frac{B_{\perp}}{B_{\perp}^2 + \nu^2} \cdot \frac{C_{\perp}}{C_{\perp}^2 + \nu^2} (1 - \cos[(\Omega_2 - \Omega_3)\Delta z]) \right. \\ &+ \frac{A_{\perp}}{A_{\perp}^2 + \nu^2} \cdot \left(\frac{D_{\perp}}{D_{\perp}^2 + \nu^2} - \frac{A_{\perp}}{A_{\perp}^2 + \nu^2} \right) (1 - \cos[\Omega_4\Delta z]) - \frac{A_{\perp}}{A_{\perp}^2 + \nu^2} \cdot \frac{D_{\perp}}{D_{\perp}^2 + \nu^2} (1 - \cos[\Omega_5\Delta z]) \\ &+ \left. \left. \frac{1}{N_c^2} \frac{B_{\perp}}{B_{\perp}^2 + \nu^2} \cdot \left(\frac{A_{\perp}}{A_{\perp}^2 + \nu^2} - \frac{B_{\perp}}{B_{\perp}^2 + \nu^2} \right) (1 - \cos[(\Omega_1 - \Omega_2)\Delta z]) \right] \right\} \\ &+ x^3 m^2 \left[\frac{1}{B_{\perp}^2 + \nu^2} \cdot \left(\frac{1}{B_{\perp}^2 + \nu^2} - \frac{1}{C_{\perp}^2 + \nu^2} \right) (1 - \cos[(\Omega_1 - \Omega_2)\Delta z]) + \dots \right] \end{aligned}$$

- Full massive in-medium splitting functions now available
- Can be evaluated numerically