# Oscillation phenomena in multiparticle production processes

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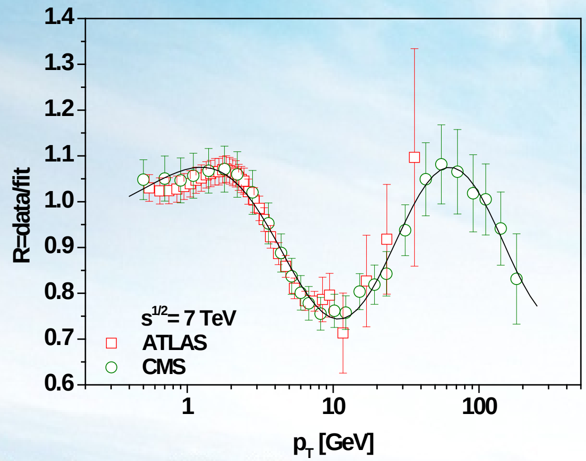
XLVI International Symposium on Multiparticle Dynamics
Seogwipo, Jeju Island, 29 August – 2 September 2016

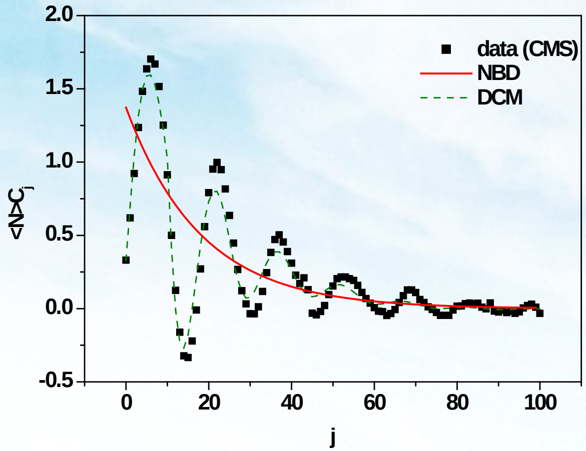
There is good evidence for the presence of oscillation in counting statistics, in many different, apparently very disparate branches of physics. Examples include:

- oscillations of the high-order cumulants of transport through a Mach-Zender interferometer , and
- in transport through a double quantum dot
- oscillations have been seen in quantum optics (in photon distribution function in slightly squeezed states)
- as well as in elementary particle physics,

further demonstrating the universality of the phenomenon in a large class of stochastic processes. In fact, whereas theoretical studies of a number of different systems have found that the high-order cumulants oscillate as functions of certain parameters, so far no systematic explanation of this phenomenon has been given.

In this presentation we concentrate on oscillation phenomena seen at LHC energies in transverse momentum distributions and multiplicity distributions.



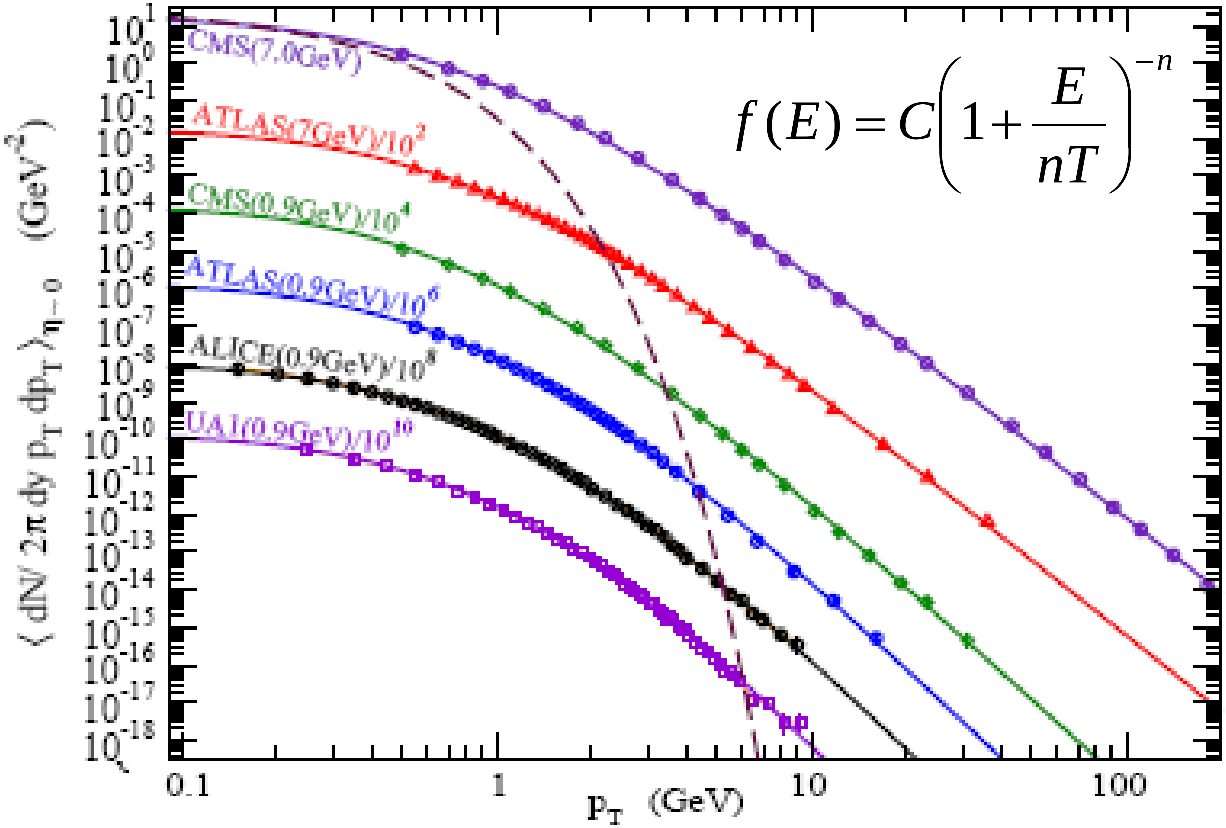




Large transverse momentum distributions apparently exhibit power-like behavior. However, we argue that, under closer inspection, this behavior is in fact decorated with some log-periodic oscillations (seen in all LHC experiments).

In what concerns multiplicity distributions P(N), they are most frequently described by the NBD. However, with increasing collision energy some systematic discrepancies become more and more apparent. The wave structure of the multiplicity distributions already observed by ALICE, CMS (and previously also by UA5) experiments is still hardly significant.

Our result is not directly connected with the wave structure observed in data on P(N) for N> 25. The coefficients Cj (connected with “combinants” ) are completely insensitive to the P(N > (j +1)) tail of the multiplicity distribution, whereas their oscillatory behavior starts from the very beginning.

# Transverse momentum distributions are characterized by a quasi-power law (Hagedorn formula or Tsallis distribution)




Transverse momenta distributions of different kinds can be described by a quasi power law formula (known as QCD-inspired Hagedorn formula or Tsallis distribution when the observation is interpreted in terms of the statistical model of particle production, employing the Tsallis non-extensive statistics) which for large values of transverse momenta becomes scale free (independent on T) power distribution $1/p_T^n$

Tsallis distribution successfully describes spectra, the flux of which changes by over 14 orders of magnitude.

C-Y Wong et al., PRD **91** (2015) 114027

# Tsallis distribution

C. Tsallis, J.Stat.Phys. **52** (1988) 479

$$\frac{2-q}{T}\left[1-(1-q)\frac{E}{T}\right]^{\frac{1}{1-q}}$$

$q \to 1$

meaning of $q$ ?

Examples of mechnisms leading to Tsallis distribution:

- Superstatistics
- Stochastic network approach
- Multiplicative noise
- MaxEnt (Shannon entropy)

**BG**

R. Hagedorn (1965)

$$\frac{1}{T}\exp\left(-\frac{E}{T}\right)$$

more information:
APPB 46 (2015) 1103
arXiv:1501.01936

# Superstatistics

Superstatistics which is a superposition of two different statistics relevant to driven nonequilibrium systems with a stationary state and intensive parameter fluctuations [C. Beck et al., Physica A322 (2003) 267]

$$h(E/T) = \int_0^\infty f(E/T)\, g(1/T)\, d(1/T)$$

## Tsallis statistics as a special case of superstatistics

$$f(E) = \frac{1}{T}\exp\left(-\frac{E}{T}\right)$$ BG

$$g(1/T) = \frac{1}{\Gamma(\frac{1}{q-1}-s)}\frac{T_0}{q-1}\left(\frac{1}{q-1}\frac{T_0}{T}\right)^{\frac{1}{q-1}-1-s}\exp\left(-\frac{1}{q-1}\frac{T_0}{T}\right)$$ gamma distr.

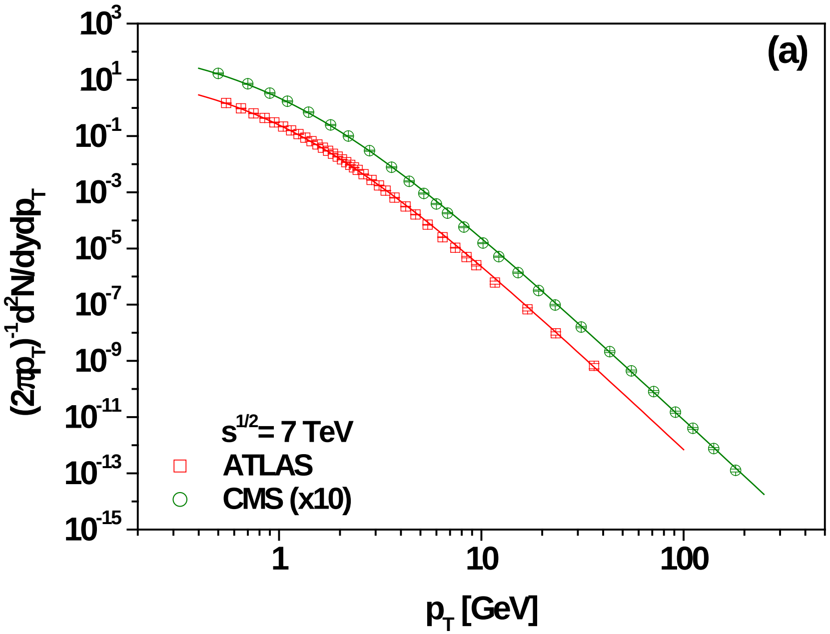
$$h_q(E) = \int_0^\infty f(E)\, g(1/T)\, d(1/T) = \frac{2-q}{T_0}\left[1-(1-q)\frac{E}{T_0}\right]^{\frac{1}{1-q}}$$ Tsallis

$$q = 1 + \frac{Var(T)}{<T>^2}$$

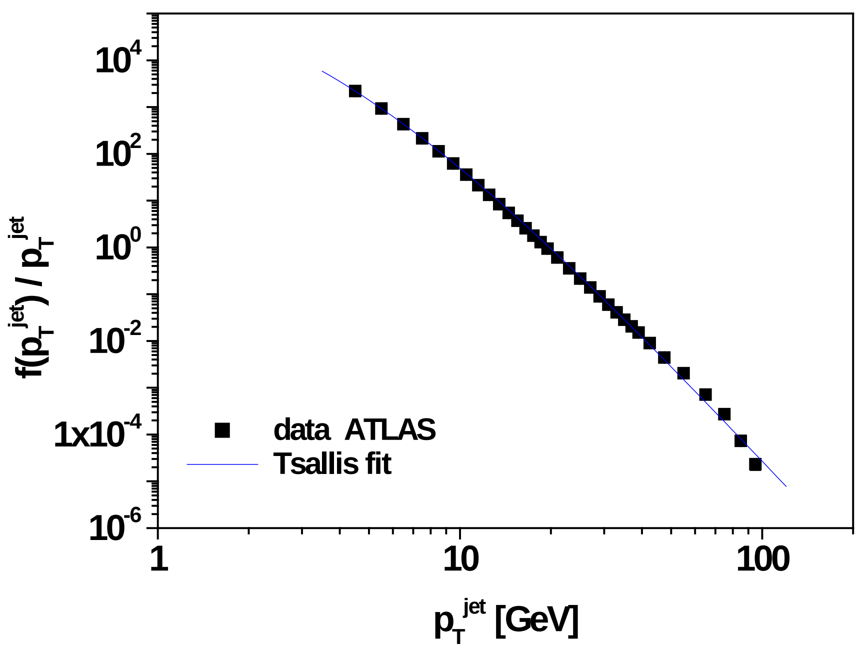
GW and ZW, PRL **84** (2000) 2770
C.Beck, PRL **87** (2001) 180601

# transverse momentum distributions are characterized by a quasi-power law (Hagedorn formula or Tsallis distribution)

hadrons



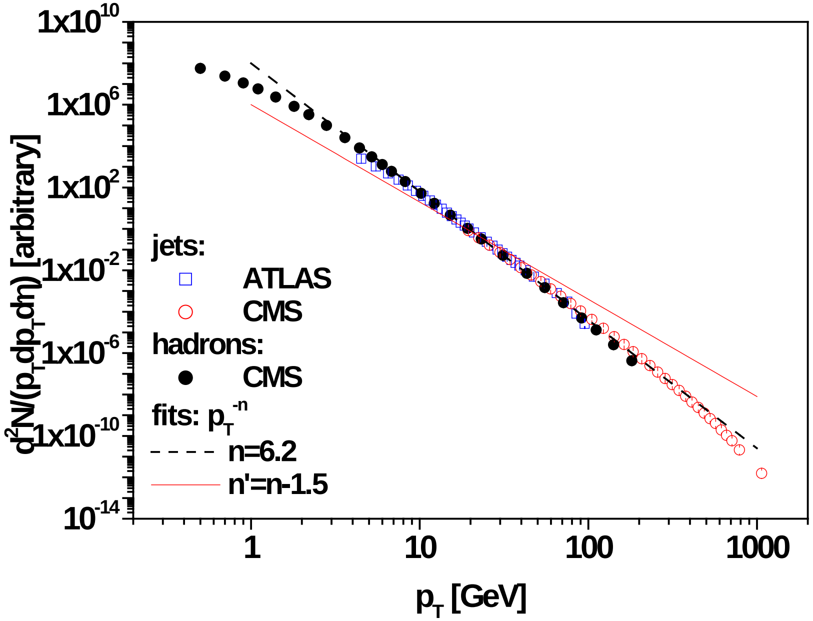

jets

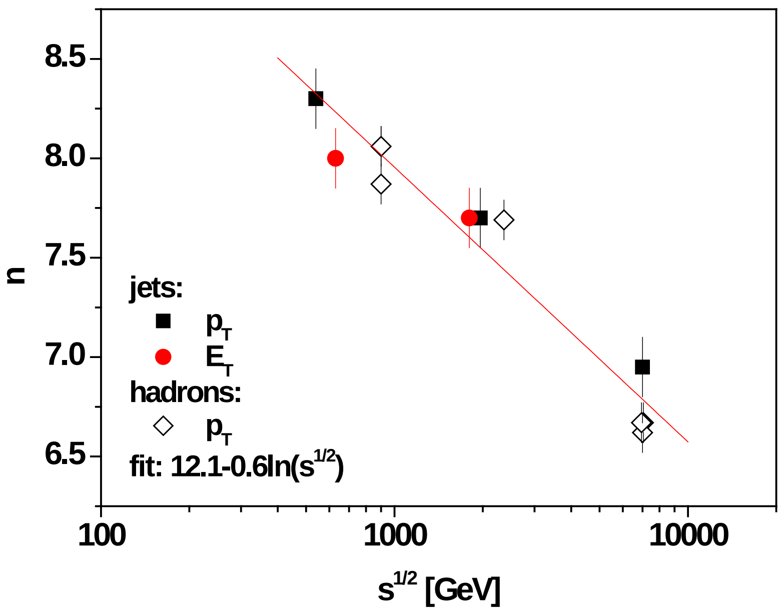



$$f(E) = C\left(1 + \frac{E}{mT}\right)^{-m}$$

$$T = 0.145\ GeV$$

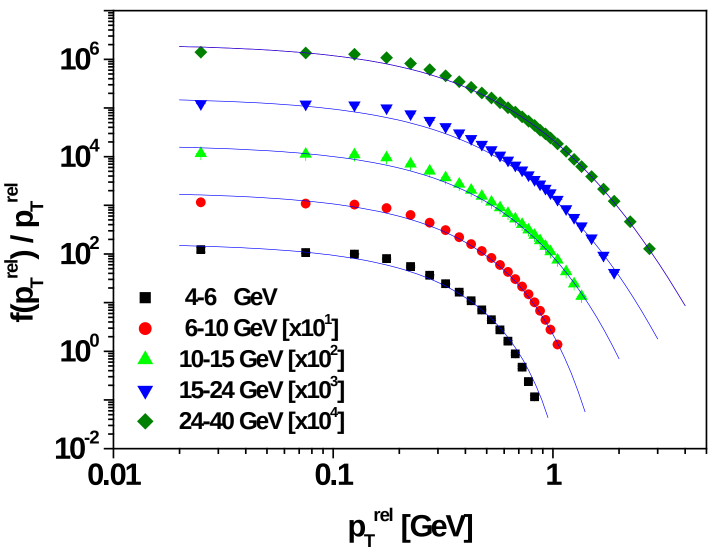
$$m = 6.7$$

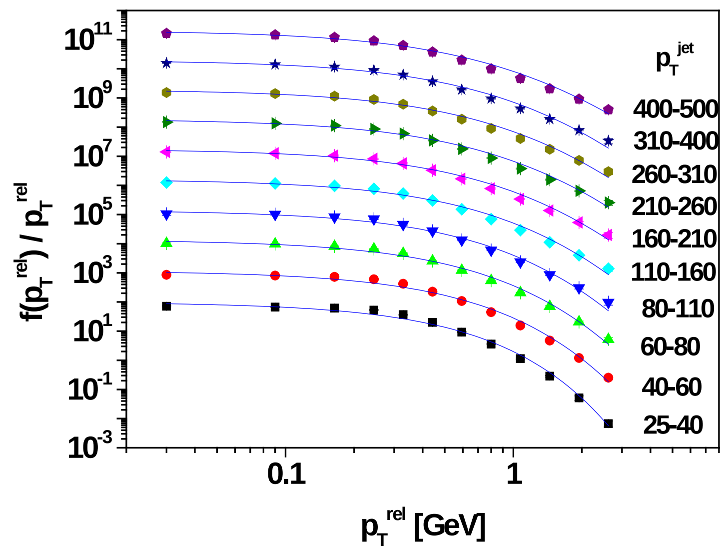


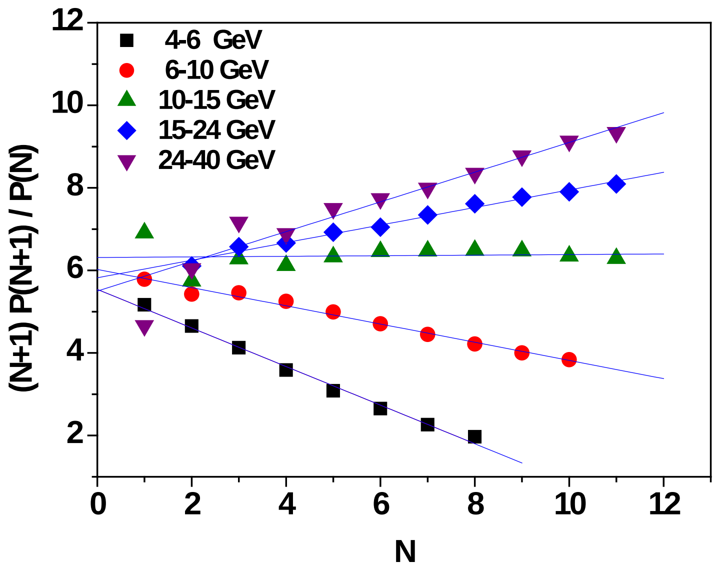

The values of the corresponding power indices are similar, strongly indicating the existence of a common mechanism behind all these processes

# Transverse momenta distribution in jet events




# Multiplicity distribution in jet events

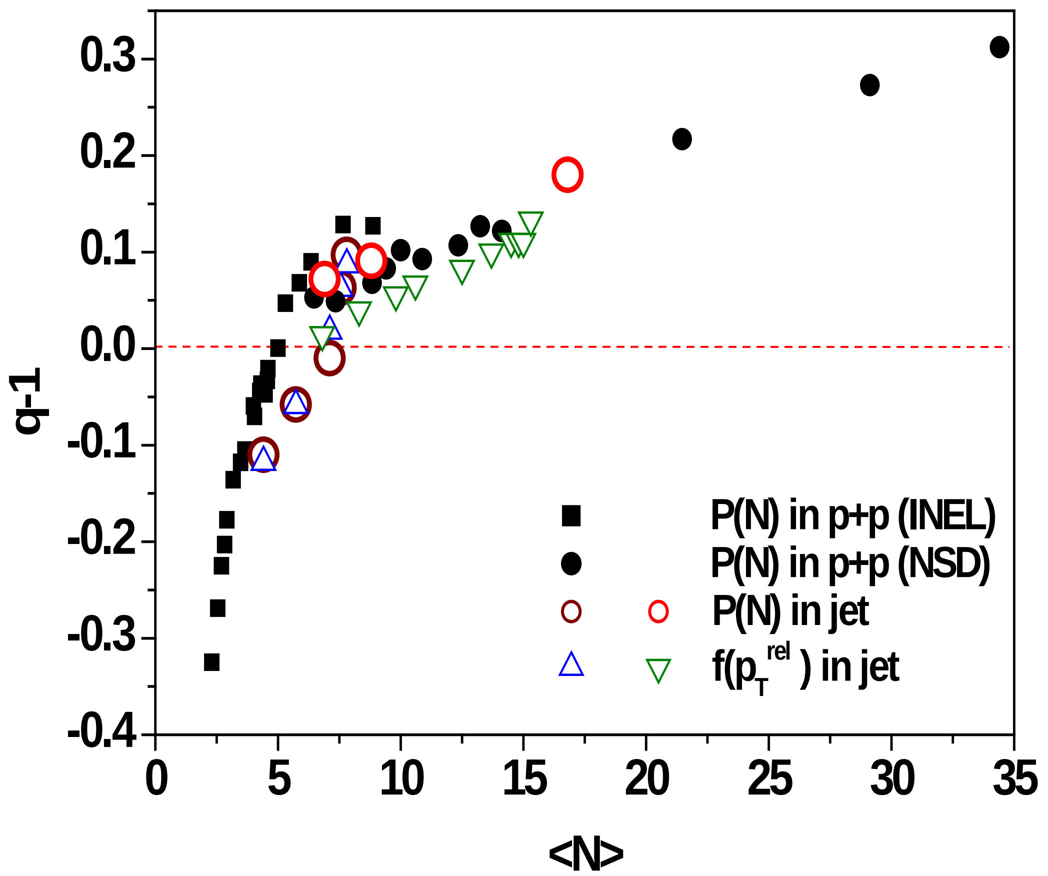



$$\frac{(N+1)P(N+1)}{P(N)} = a + bN$$

| | | | |
|---|---|---|---|
| NBD: | $a = \langle N \rangle k/(k + \langle N \rangle)$ | $b = a/k$ | $1/k = q - 1$ |
| Poisson: | $a = \langle N \rangle$ | $b = 0$ | |
| BD: | $a = \langle N \rangle \kappa/(\kappa - \langle N \rangle)$ | $b = a/\kappa$ | $1/\kappa = 1 - q$ |

[ATLAS, 7 TeV, |y|<1.9, $R = \sqrt{\Delta\eta^2 + \Delta\varphi^2}$ = 0.6]

# Self-similarity in jet events following from p-p collisions at LHC




The self-similarity of the scattering process was already recognized by Hagedorn [R. Hagedorn and R. Ranft, Suppl. Nuovo Cim. 6, 169 (1968)], who described the various possible particle states as a 'fireball' and who defined a fireball as follows:

**A fireball is**

***... a statistical equilibrium of an undetermined number of all kinds of fireballs, each of which in turn is considered to be...**

**(back to *)**

Clearly, nowadays we would call this a self-similarity assumption.

Also [G. Gustafson and A. Nilsson, Nucl. Phys. **B355**, 106, (1991)]

**QCD predicts that parton fragmentation into final state hadrons proceeds through multiple sub-jet production. This cascade of jets to sub-jets to sub-sub-jets (et cetera) to final state hadrons should demonstrate self-similar behavior.**

and [J.D. Bjorken, Phys. Rev. **D45**, 4077 (1992)]

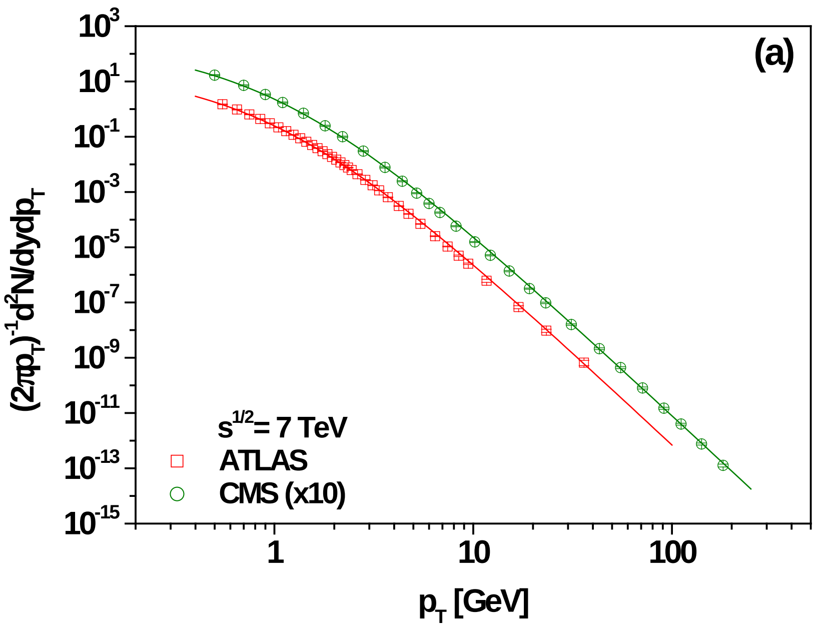
**In QCD extra gluons of lower-pt, scales can also be radiated. This provides new populations of jets, which again extend the entire lego plot, including the extensions we have exhibited. The self-similar character of this extension should be evident.**

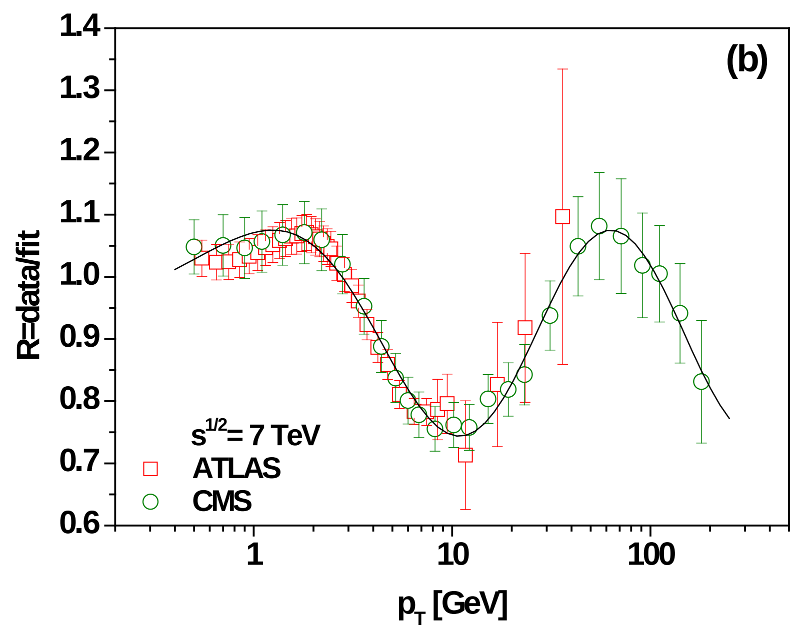
GW and ZW, Phys.Lett.B **727** (2013) 163 [arXiv:1310.0671]

# Tsallis distribution decorated with log-periodic oscillation

**transverse momentum distributions are characterized by a quasi-power law (Tsallis distribution)**

**Tsallis distribution is decorated with log-periodic oscillations**



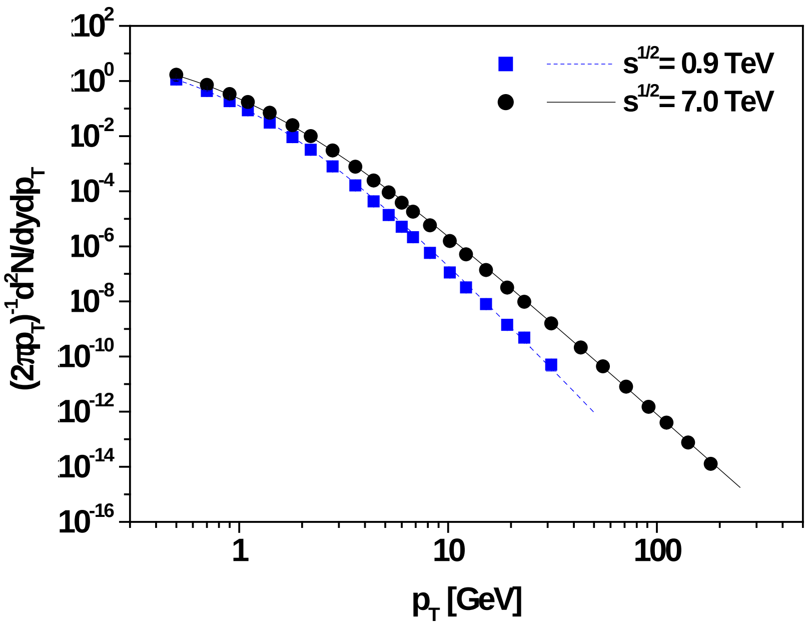

$$f(E) = C\left(1 + \frac{E}{mT}\right)^{-m}$$

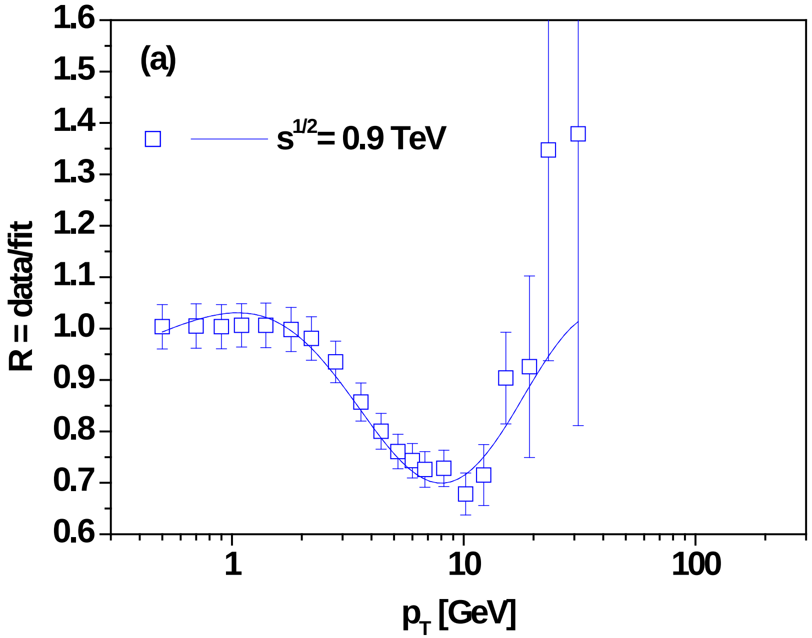
$$T = 0.145\,GeV \qquad m = 6.7$$

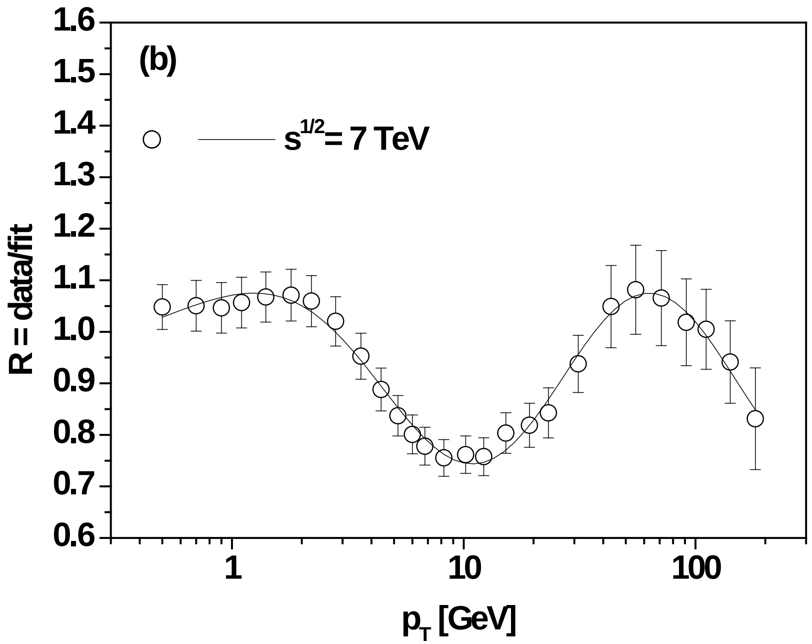
R(E) = a + b cos [c ln(E + d) + f]

a = 0.909, b = 0.166, c = 1.86, d = 0.948 and f = −1.462

# Different energies

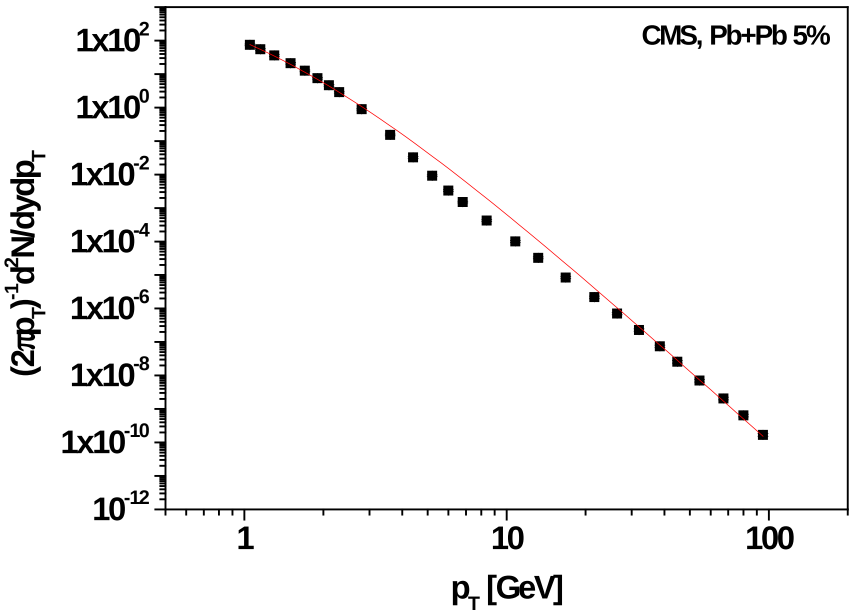


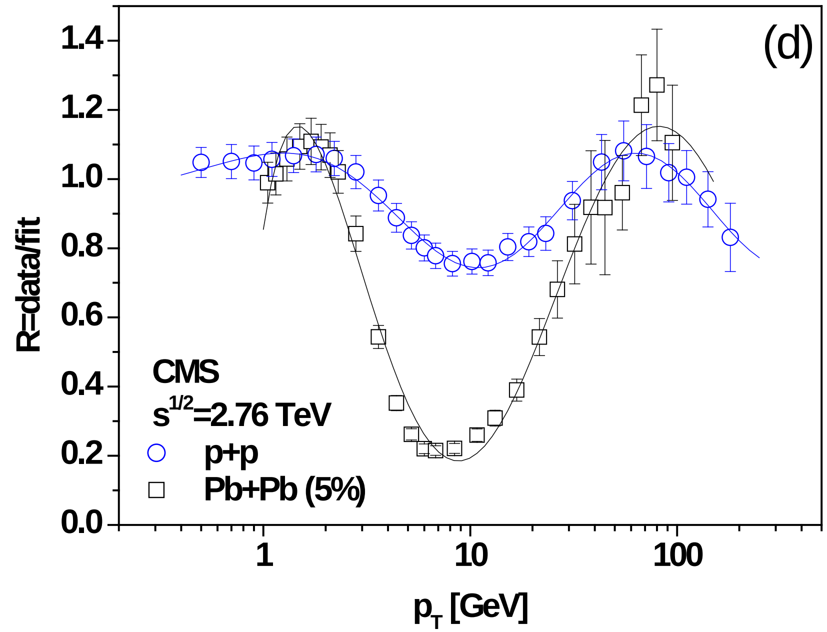



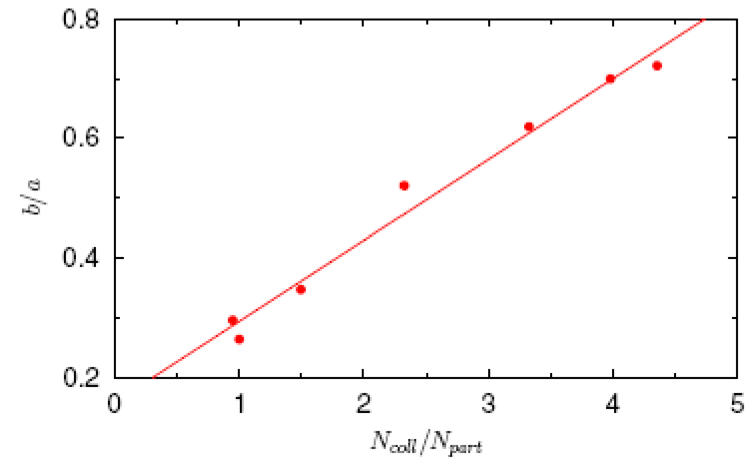

Fit to *data for pp collisions at 0.9 and 7* TeV from CMS experiment.
Parameters used are, respectively,
(*T* = *0.135, m = 8) and (T = 0.145, m = 6.7).*

# Different collision systems




Fit to *data for Pb+Pb collisions (5% centrality) at 2.76* TeV from CMS experiment.
Parameters used are $T = 0.15$, $m = 7.05$



# Scale invariance

if for some function O(x), one finds that

$$O(\lambda x) = \mu O(x)$$

then it is scale invariant and its form follows a simple power law,

$$O(x) = Cx^{-m}$$ with $$m = -\ln \mu / \ln \lambda$$

This relation can be written as

$$\mu\lambda^{m} = 1 = e^{i2\pi k}$$

where k is an arbitrary integer. It means therefore that, in general,

$$m = -\ln \mu / \ln \lambda + i2\pi k / \ln \lambda,$$

i.e., it is a complex number, the imaginary part of which signals a hierarchy of scales leading to

**Log-periodic oscillations**

$$\frac{df(E)}{dE} = -\frac{1}{T} f(E)$$

BG distribution

$$f(E) = \frac{1}{T}\exp\left(-\frac{E}{T}\right)$$

If the scale parameter is dependent on variable (preferential attachment)

$$T = T(E) = T_0 + (q-1)E$$

$$\frac{df(E)}{dE} = -\frac{1}{T(E)} f(E) = -\frac{1}{T_0 + (q-1)E} f(E)$$

Tsallis distribution

$$n = 1/(q-1)$$

$$f(E) = \frac{n-1}{nT_0}\left(1 + \frac{E}{nT_0}\right)^{-n}$$

dE → $\delta E$ finite

$$f(E+\delta E) = \frac{-n\delta E + nT + E}{nT + E} f(E)$$

$$\delta E = \alpha(nT + E)$$

$$f(E + \alpha(nT + E)) = (1 - \alpha n) f(E)$$

scale invariant relation

$$x = (1 + \frac{E}{nT})$$

$$g((1+\alpha)x) = (1 - \alpha n) g(x)$$

$$g((1+\alpha)x) = (1-\alpha n)g(x)$$

its solution is power law

$$g(x) \sim x^{-m_k}$$

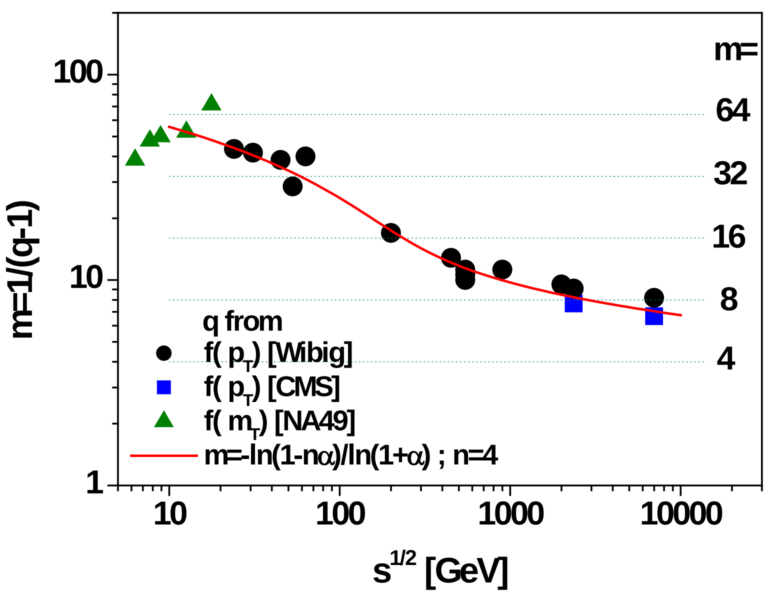
with exponent $m_k$ depending on $\alpha$ and acquiring an imaginary part

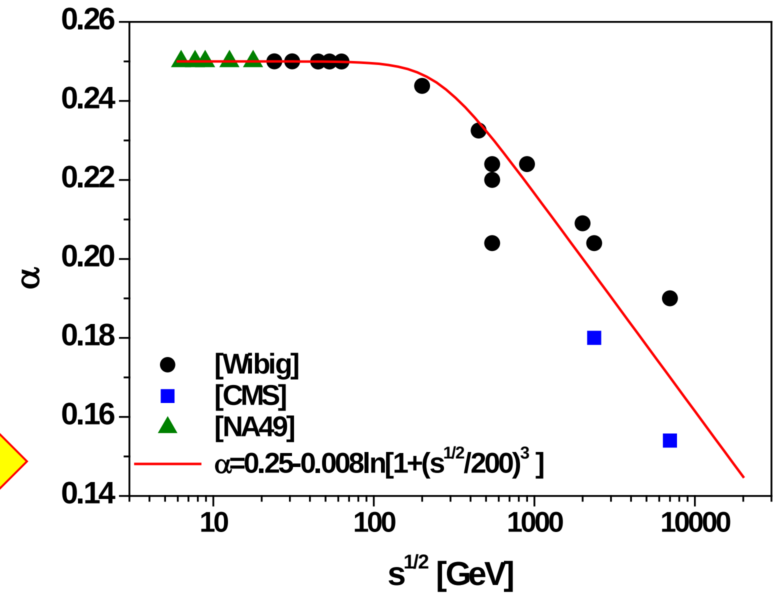
$$m_k = -\frac{\ln(1-\alpha n)}{\ln(1+\alpha)} + ik\frac{2\pi}{\ln(1+\alpha)}$$

In the special case (real solution) $k=0$ the power $m_0$ still depends on $\alpha$ and increases with it roughly as

$$m_0 \cong n + \left(\frac{n}{2}(n+1)\right)\alpha + \left(\frac{n}{12}(4n^2+3n-1)\right)\alpha^2 + \left(\frac{n}{24}(6n^3+4n^2-n+1)\right)\alpha^3 \qquad \alpha < 1/n$$

recovers in the limit $\alpha \to 0$ the power $n$ in the usual Tsallis distribution

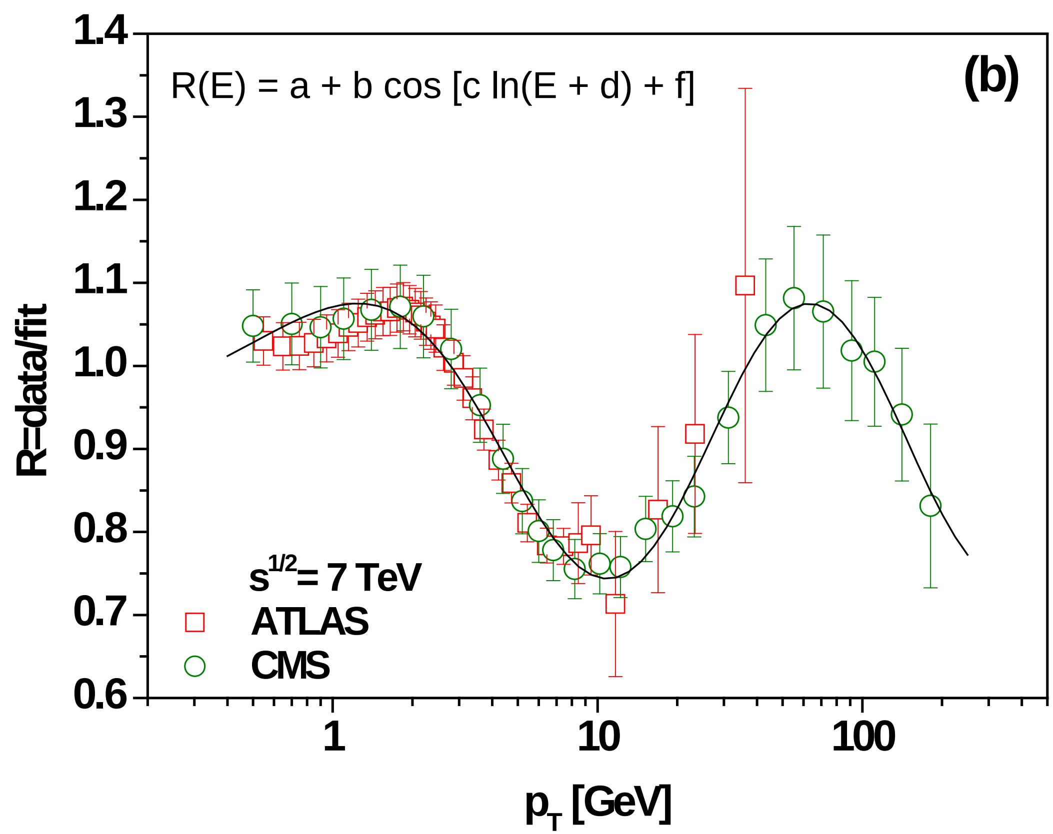


$$g(x) = \sum_k w_k \,\mathrm{Re}(x^{-m_k}) = x^{-\mathrm{Re}(m_k)} \sum_k w_k \cos[\mathrm{Im}(m_k)\ln(x)]$$

$$g(E) = \left(1+\frac{E}{nT}\right)^{-m_0} \left\{ w_0 + w_1 \cos\left[\frac{2\pi}{\ln(1+\alpha)} \ln\left(1+\frac{E}{nT}\right)\right]\right\}$$

$$g(E) = \left(1+\frac{E}{nT}\right)^{-m_0} R(E)$$



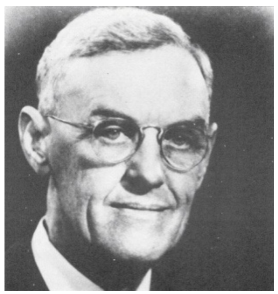

$$a/b = w_0/w_1$$

$$c = 2\pi/\ln(1+\alpha)$$

$$d = nT$$

$$f = -2\pi\ln(nT)/\ln(1+\alpha)$$

S. D. Poisson (1781-1840)



G. W. Snedecor (1881-1974)

V. Pareto (1848-1923)
K. S. Lomax

C.Tsallis, 1988

$$f(x) \propto \exp\left(-\frac{x}{\sigma}\right)$$

$n \to \infty$

$$f(x) \propto \left(1+\frac{2x}{n}\right)^{-\frac{1}{2}(n+2)}$$

$n \in \mathrm{N}$

$$f(x) \propto \left(1+\beta\frac{x}{n}\right)^{-n}$$

$n \in \mathrm{R}$

$\mathrm{Im}\, n \to 0$

**Log-periodic oscillations**

$n \in \mathrm{C}$

**?**

# Superstatistics

$$f(x) \propto \exp\left(-\frac{x}{\sigma}\right)$$

$$f(x) \propto \left(1+\frac{x}{\gamma}\right)^{-2}$$

$n \to \infty$ $\qquad$ $n \to 2$

$g(1/T)$ $\chi^2$ distr.

C.Beck, PRL **87** (2001) 180601

$$f(x) \propto \left(1+\frac{2x}{n}\right)^{-\frac{1}{2}(n+2)} \qquad n \in N$$

$g(1/T)$ gamma distr.

GW and ZW, PRL **84** (2000) 2770

$$f(x) \propto \left(1+\beta\frac{x}{n}\right)^{-n} \qquad n \in R$$

$\mathrm{Im}\, n \to 0$

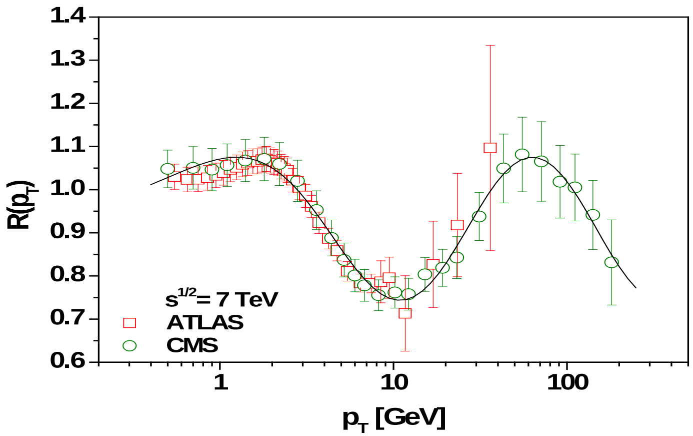
Log-periodic oscillations **?** $\qquad n \in C$

Superstatistics is a superposition of two different statistics relevant to driven nonequilibrium systems with a stationary state and intensive parameter fluctuations

$$h(E/T) = \int_0^\infty f(E/T)\, g(1/T)\, d(1/T)$$

# Log-periodic oscillations

$$g(E) = C\left(1+\frac{E}{nT}\right)^{-m_0} R(E)$$




- results in complex nonextensivity parameter $q = q' + iq''$
- also the heat capacity $C = 1/(q-1)$ becomes complex: $C = C' - iC'' = C_\infty + \frac{C_0 - C_\infty}{1+(\omega\tau)^2}(1 - i\omega\tau)$

  A complex $C_V(\omega, k)$ means that $\delta U$ and $\delta T$ are shifted in phase and that the entropy production in the system differs from zero.

  $$q - 1 = \frac{Var(T)}{<T>^2} - i\frac{S(T)}{<T>^2}$$

  where the spectral density of temperature fluctuations $S(T) = \omega \int < Cov[T(0), T(t)] > e^{-i\omega t} dt$

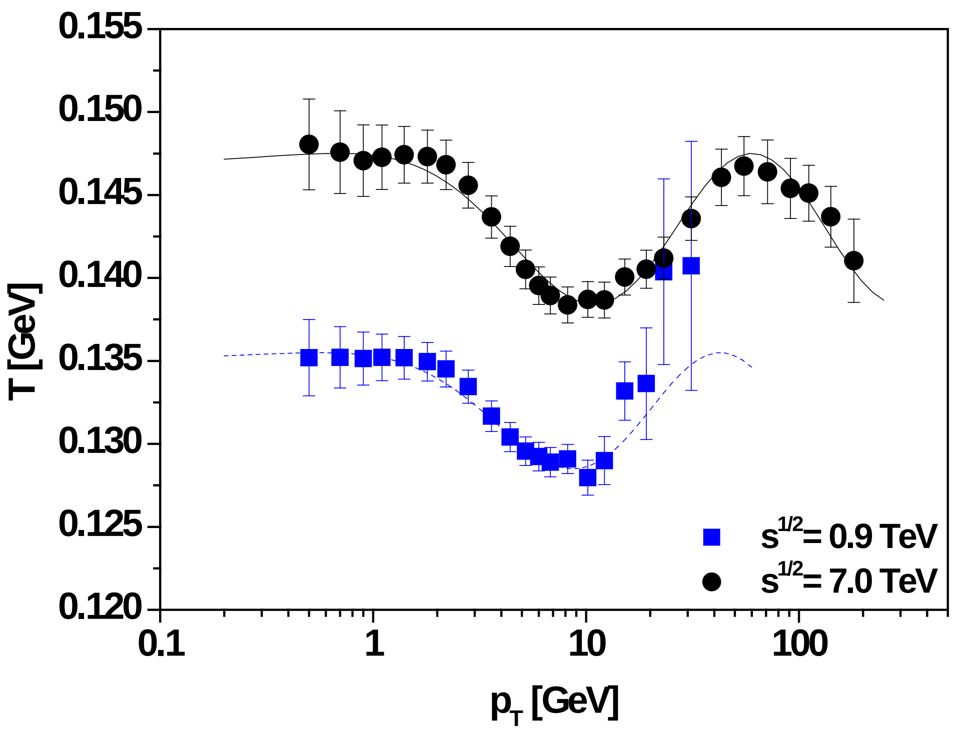
- complex multiplicative noise

  in Langevin equation $dp/dt + \gamma(t) p = \xi(t)$ $\quad \gamma(t) = \gamma_0(t) + i\gamma_1$ $\quad E(\gamma)$ complex

- the complex pdf

  the imaginary part is proportional to the degree of incompatibility of the correlated stochastic processes.

# log-periodic scale parameter T




$$T = \bar{a} + \bar{b}\sin[\bar{c}(\ln(E+\bar{d}) + \bar{f}]$$

Stochastic equation for the temperature evolution in Langevin formulation with energy dependent noise $\xi(t,E)$

$$\frac{dT}{dt} + \frac{1}{\tau}T + \xi(t,E)T = \Phi$$

leads to

$$\frac{1}{n}\frac{d^2T}{d(\ln E)^2} + \left[\frac{1}{\tau} + \xi(t,E)\right]\frac{dT}{d(\ln E)} + T\frac{d\xi(t,E)}{d\ln E} = 0$$

which for the noise

$$\xi(t,E) = \xi_0(t) + \frac{\omega^2}{n}\ln E$$

(and relaxation time $\tau = const$ )

is just an equation for the damping harmonic oscillator and has a solution

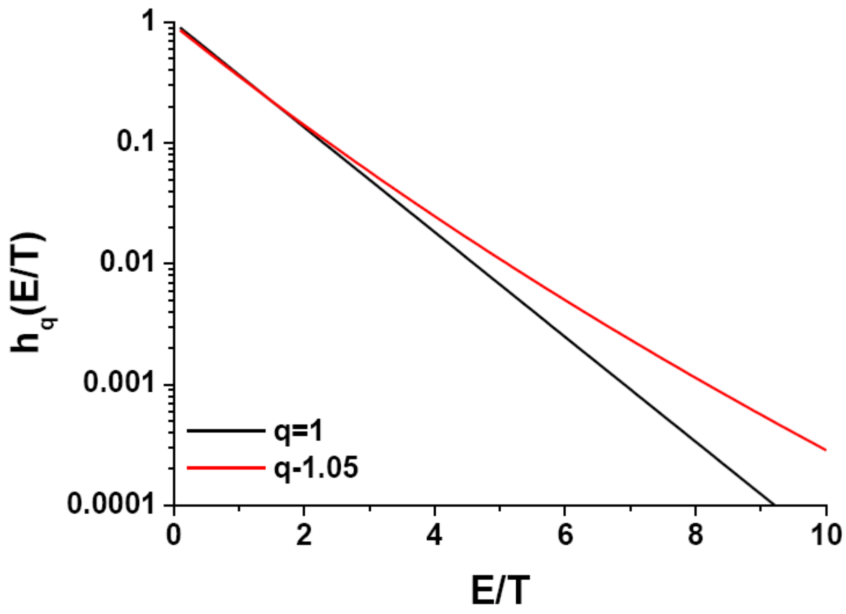
$$T = C\exp\left\{-n\cdot\left[\frac{1}{2\tau} + \frac{\xi(t,E)}{2}\right]\ln E\right\}\cdot\sin(\omega\ln E + \phi)$$

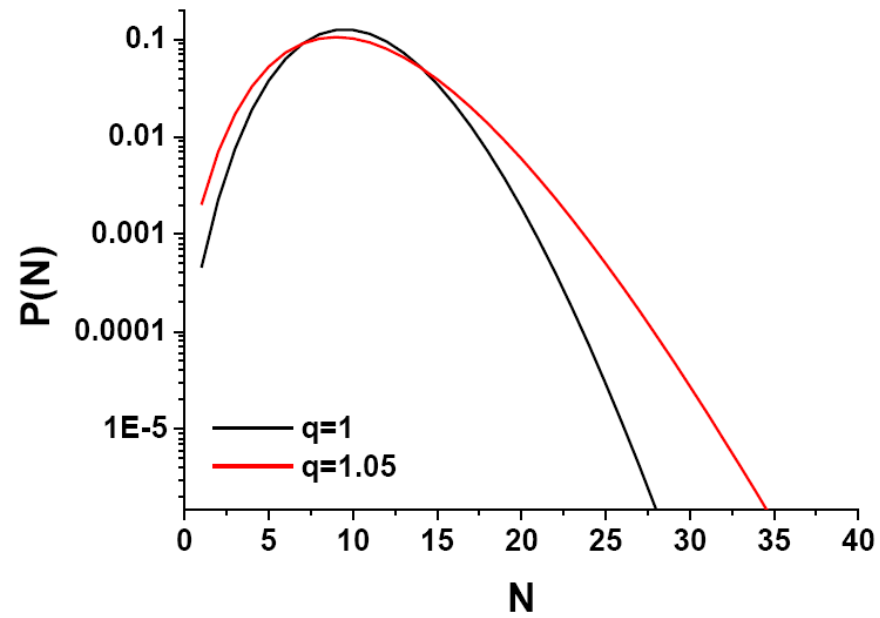
We could equivalently assume the energy independent noise, $\xi(t,E) = \xi_0(t)$

but allow for the energy dependent relaxation time

$$\tau = \tau(E) = \frac{n\tau_0}{n + \omega^2\ln E}$$

# Temperature fluctuations vs. multiplicity fluctuations




$$g(E_i) = C\exp\left(-\frac{E_i}{T}\right) \quad \Rightarrow$$

$$h_q(E_i) = C_q\left[1-(1-q)\frac{E_i}{T_{eff}}\right]^{\frac{1}{1-q}}$$

*where* $q = 1 + \frac{Var(T)}{\langle T\rangle^2}$

*and* $T_{eff} = T_0 + (q-1)T_{visc}$

$$\sum_{i=0}^{N} E_i \le E \le \sum_{i=0}^{N+1} E_i$$

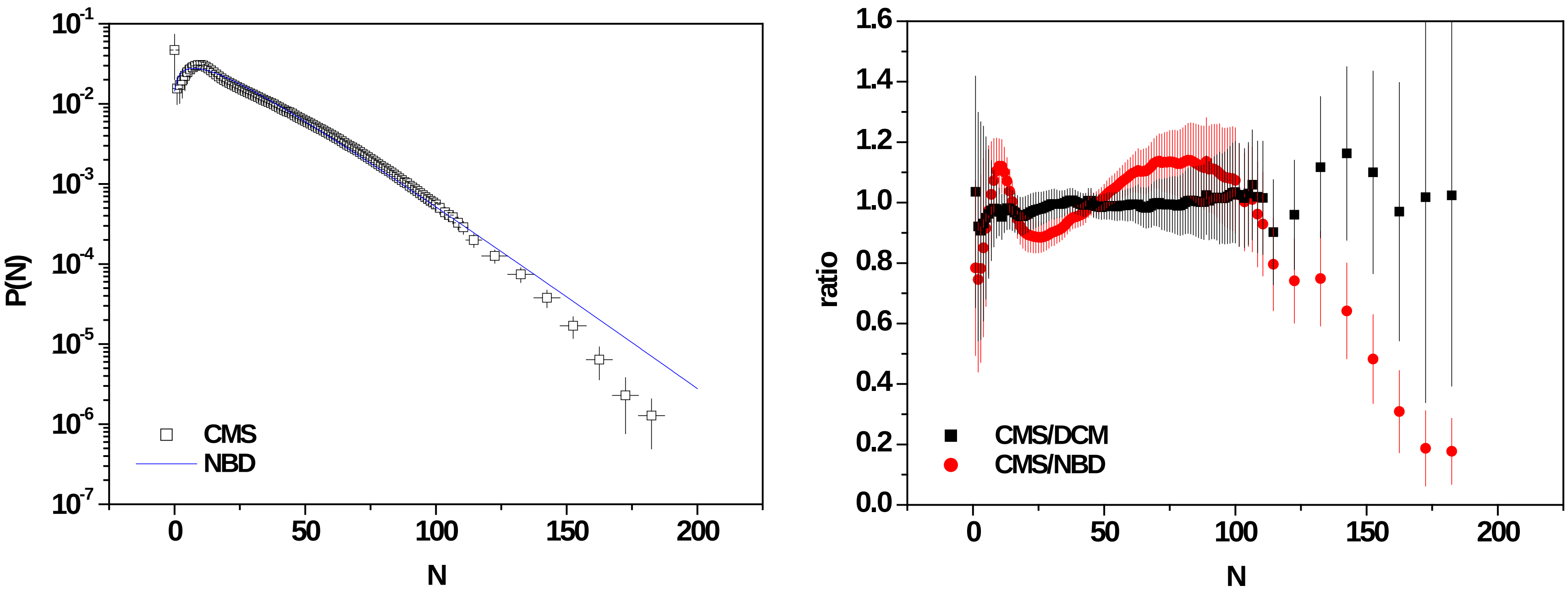
$$P(N) = \frac{\langle N\rangle^N}{N!}\exp(-\langle N\rangle); \quad \langle N\rangle = \frac{E}{T} \quad \Rightarrow$$

$$P(N) = \frac{\Gamma(N+k)}{\Gamma(N+1)\Gamma(k)} \frac{\left(\frac{\langle N\rangle}{k}\right)^N}{\left(1+\frac{\langle N\rangle}{k}\right)^{N+k}}$$

*where* $k = \frac{1}{q-1}$ *and* $Var(N) = \langle N\rangle + (q-1)\langle N\rangle^2$


G.Wilk, Z.Włodarczyk, PRC79(2009)054903; EPJA40(2009)299; Physica A376(2007)279

# multiplicity distributions



# Recurence relation

$$(N+1)P(N+1) = g(N)P(N)$$

$$g(N) = \alpha + \beta N$$

| | | |
|---|---|---|
| NBD | $\alpha = \frac{<N>k}{k+<N>}$ | $\beta = \frac{\alpha}{k}$ |
| Poisson | $\alpha = <N>$ | $\beta = 0$ |
| BD | $\alpha = \frac{<N>\kappa}{\kappa-<N>}$ | $\beta = \frac{\alpha}{\kappa}$ |

For $N>0$ the function $g(N)$ does not depend on the acceptance $\bar{\alpha}$ (acceptance process is described by BD).

$$g(N) = \frac{(N+1)P(N+1)}{P(N)} = \frac{\left.\frac{d^{N+1}G(s)}{ds^{N+1}}\right|_{s=0}}{\left.\frac{d^{N}G(s)}{ds^{N}}\right|_{s=0}}$$

However for $N=0$

$$g(0) = \frac{\bar{\alpha}}{1-\bar{\alpha}+\bar{\alpha}\,G_M(0)} \left.\frac{dG_M(s)}{ds}\right|_{s=0}$$

$$(N+1)P(N+1) = \langle N\rangle \sum_{j=0}^{N} C_j P(N-j)$$

$$\sum_{j=0}^{\infty} C_j = 1$$

The coefficients Cj tell us how P(N+1) depends on P(N −j), i.e., they encode the memory about particles produced earlier. In the case of the NBD, this memory exponentially disappears with increasing distance (rank) j.

$$\langle N\rangle C_j = (j+1)\left[\frac{P(j+1)}{P(0)}\right] - \langle N\rangle \sum_{i=0}^{j-1} C_i \left[\frac{P(j-i)}{P(0)}\right]$$

$$C_j = \frac{(j+1)}{\langle N \rangle} C^{\star}_{j+1}$$

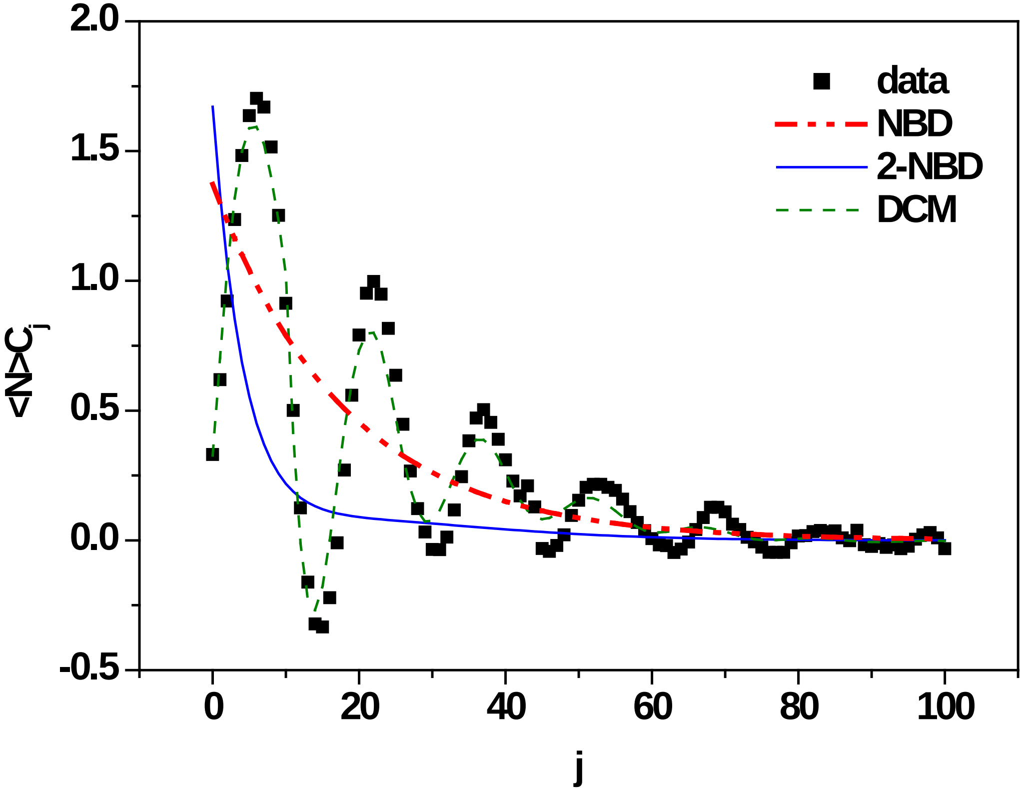
$$C^{\star}_j = \frac{1}{j!} \frac{d^j \ln G(z)}{dz^j}\Big|_{z=0}$$

**combinants**

S. Hegyi, Phys. Lett. B 463 (1999) 126
S.K.Kauffman, M.Gyulassy, JPA11 (1978)1715

$$\langle N \rangle C_j = (j+1)\left[\frac{P(j+1)}{P(0)}\right] - \langle N \rangle \sum_{i=0}^{j-1} C_i \left[\frac{P(j-i)}{P(0)}\right]$$

**Knowing P(N) one can obtain the coefficients Cj using this recurrence formula**



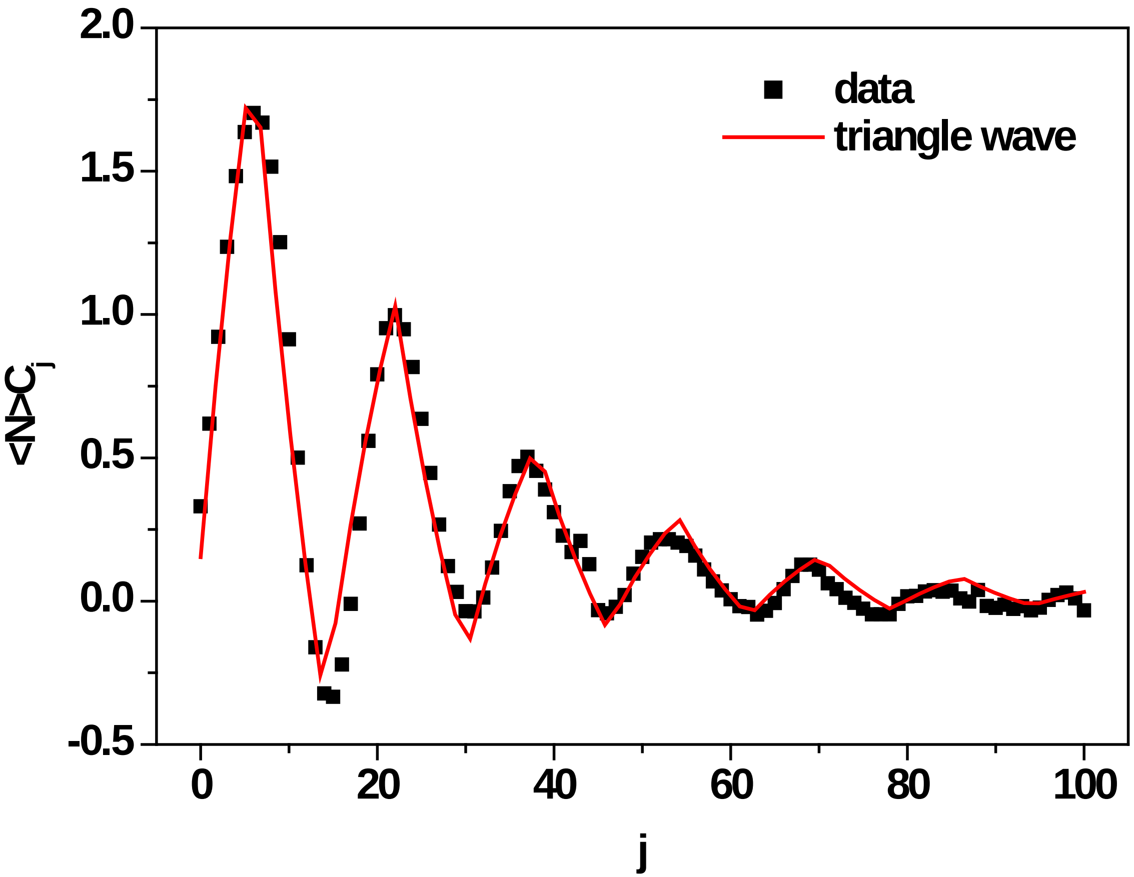

**smooth dependence on rank j for NBD**

$$C_j = \frac{k}{\langle N \rangle} p^{j+1} = \frac{k}{k+m} \exp(j \ln p)$$

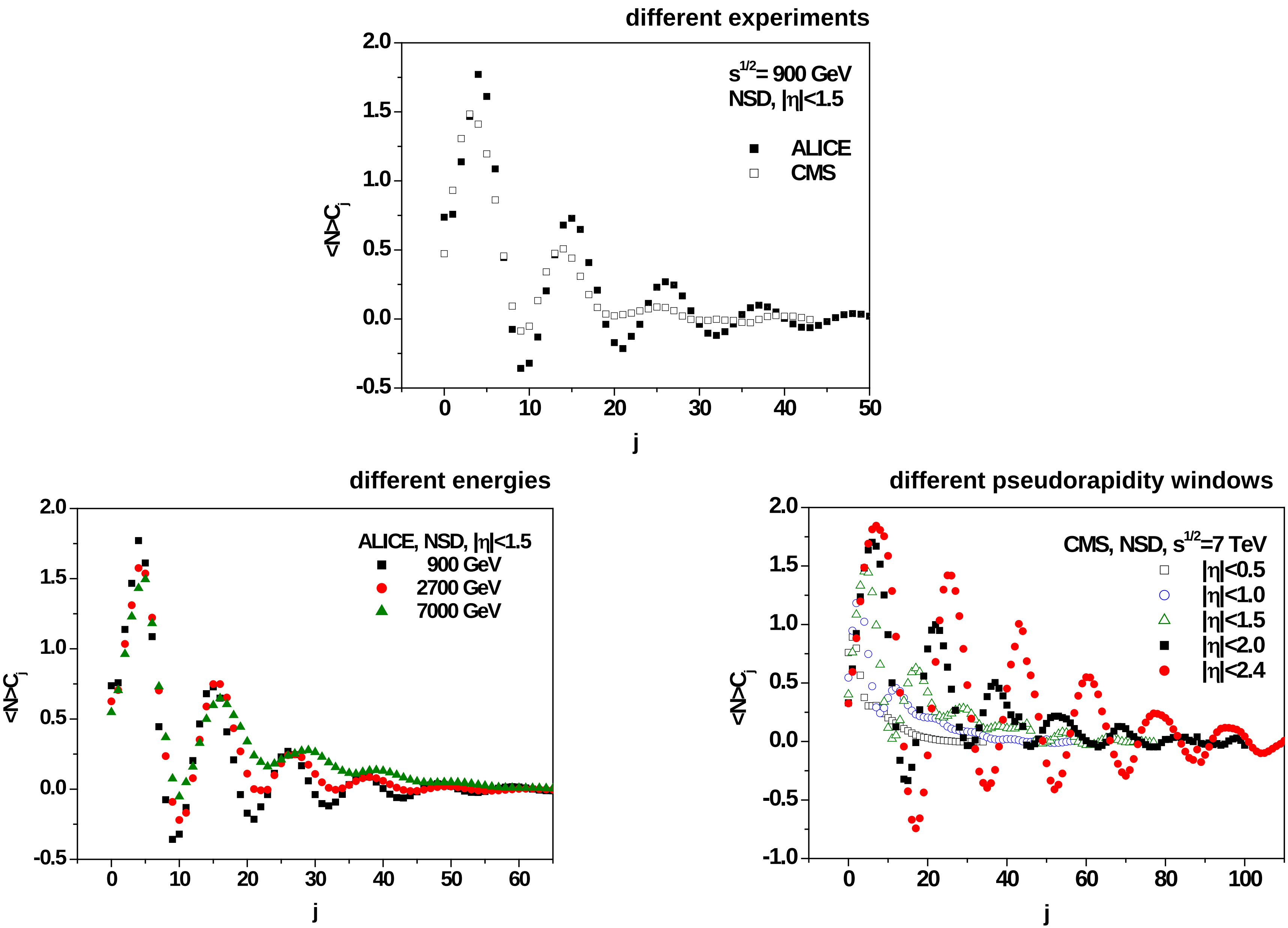
Coefficients , Cj fitted by a triangular wave, $C_j \propto (2/\pi)\,\mathrm{arcsin}[\sin(2\pi j/\omega)]$

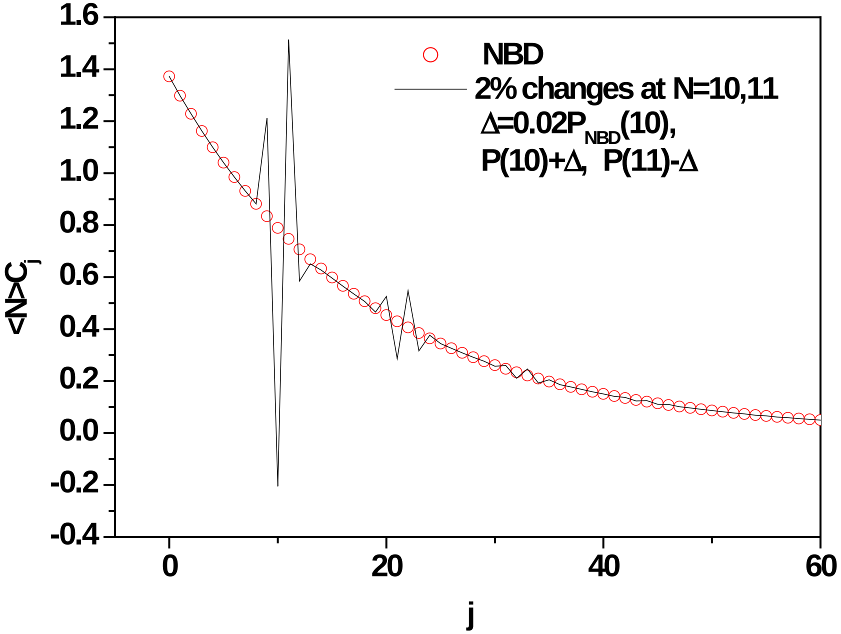
damped exponentially by some exponential factor $\propto \exp(-j/\lambda)$

$$\langle N\rangle C_j = \left\{ a_1 \left[ 1 - \left| 1 - 2\left( \frac{j}{\omega} - Int\left(\frac{j}{\omega}\right)\right)\right| \right] - a_2 \right\} \cdot \exp\left(-\frac{j}{\lambda}\right)$$



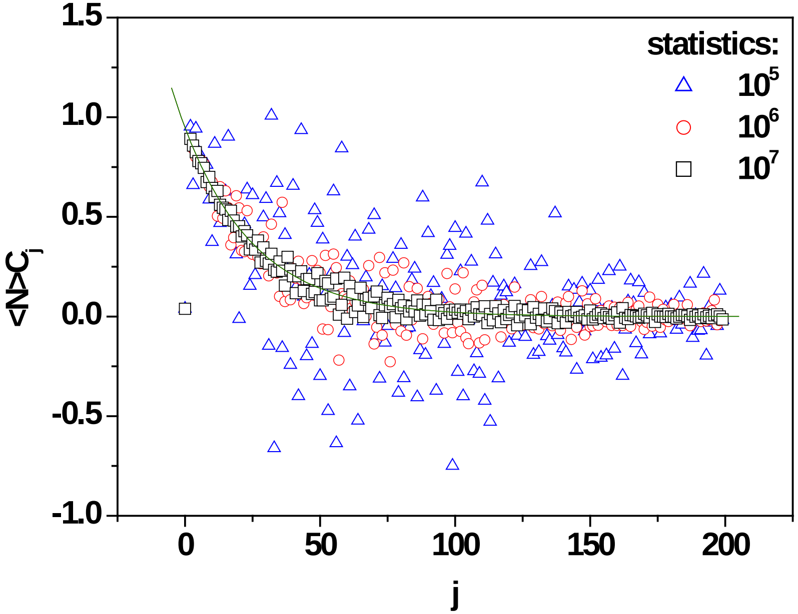
# Cj emerging from the experimental data

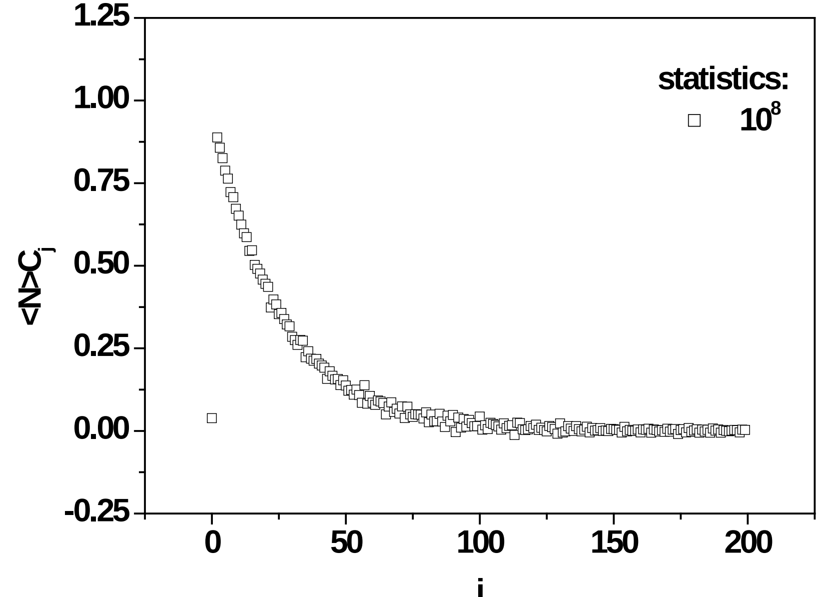





coefficients Cj are very sensitive to all changes in the original P(N)

coefficients Cj are insensitive to the statistics



# Dynamical Clan Model (DCM)

NBD

$$P(N) = \frac{\Gamma(N+k)}{\Gamma(N+1)\Gamma(k)} p^N (1-p)^k$$

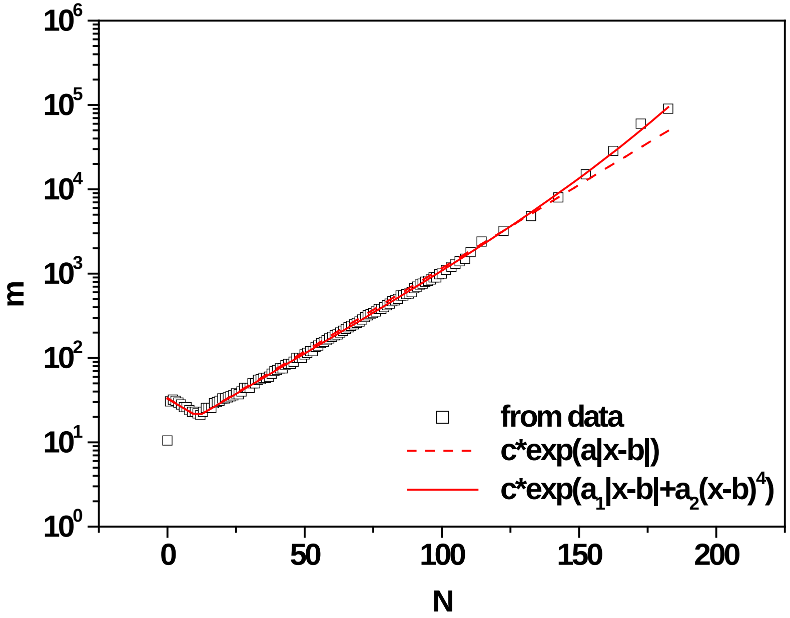
$$p = \frac{m}{m+k}$$

$$m = const$$

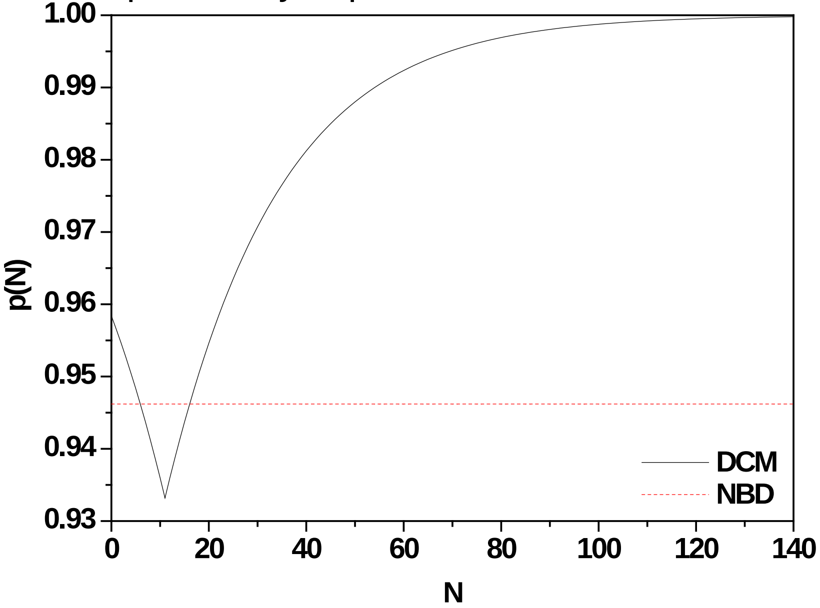
DCM

$$p(N) = \frac{1}{1 + \frac{c}{k}\exp(a|N-b|)}$$

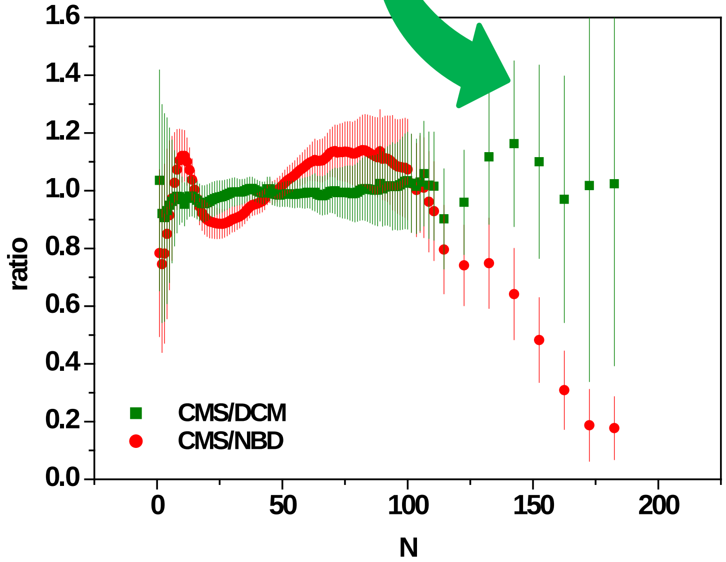
$$m = m(N) = c\exp(a|N-b|)$$

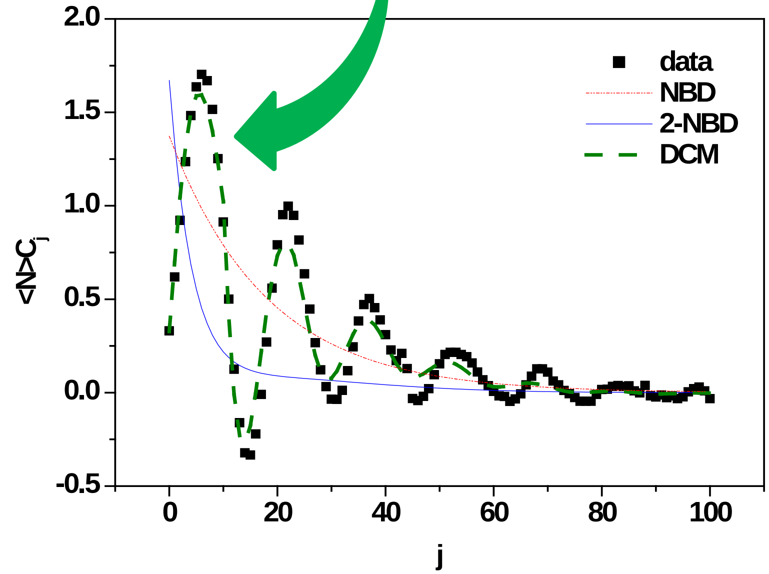



probability of particle emissiom



# Results from DCM



# Concluding Remarks

- Transverse momentum distributions are characterized by a quasi-power law (Tsallis distribution) decorated with log-periodic oscillations.

- Log-periodic structures in the data indicate that the system and/or the underlying physical mechanisms have characteristic scale invariance behavior. The discrete scale invariance and its associated complex exponents can appear spontaneously, without a pre-existing hierarchical structure.

- Tsallis distribution in energy results in NBD for multiplicity distribution.

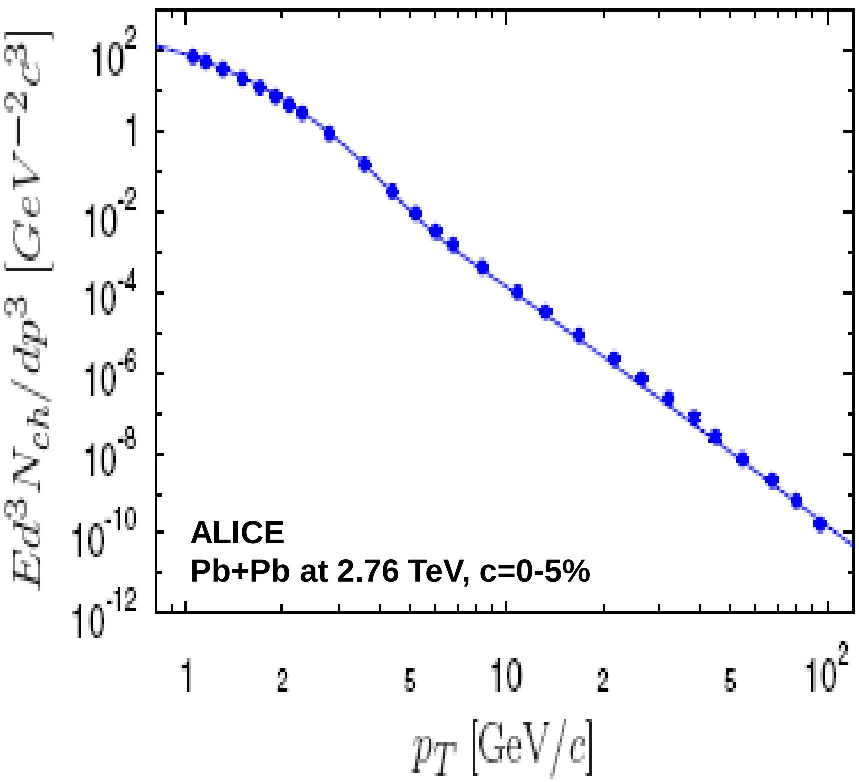
- We observe strong oscillations of coefficients Cj at LHC energies.
  The coefficients Cj tell us how P(N+1) depends on P(N −j), i.e., they encode the memory about particles produced earlier. For the NBD this memory exponentially disappears with increasing distance (rank) j.

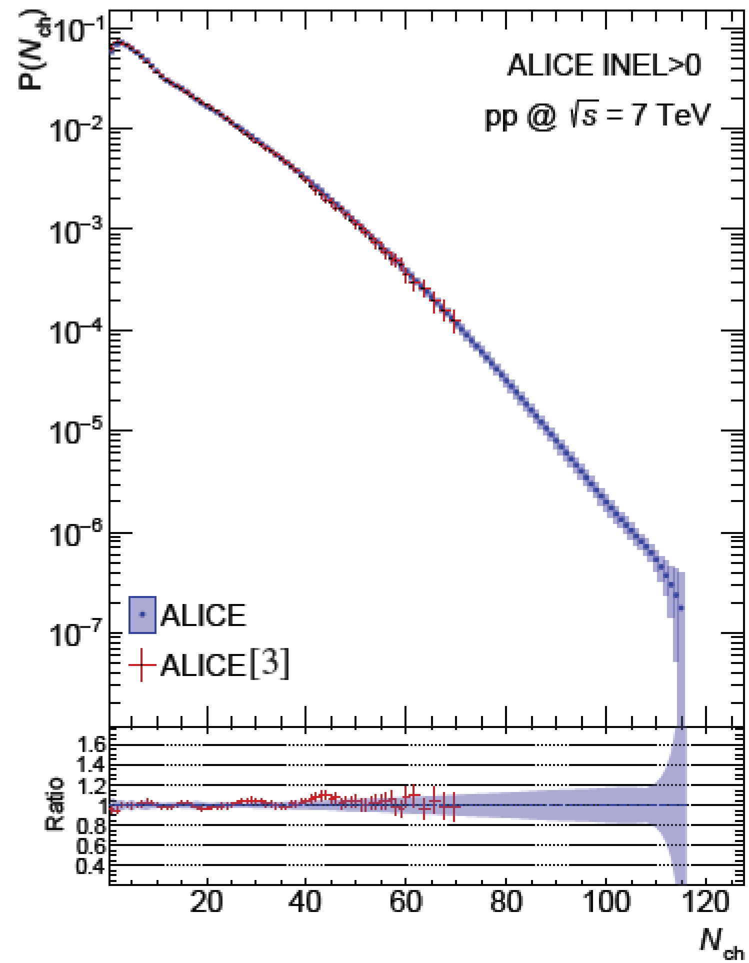
- The coefficients Cj are completely insensitive to the P(N > (j +1)) tail of the multiplicity distribution.
  Our analysis is not directly connected with the wave structure observed in data on P(N) for multiplicities N > 25. The oscillatory behavior of Cj is observed starting from the very beginning.

# Alternatives? - Two-component models

$$h(p_T) = \alpha_1\left(1+\frac{p_T}{m_1 T_1}\right)^{-m_1} + \alpha_2\left(1+\frac{p_T}{m_2 T_2}\right)^{-m_2}$$

$$P(N) = \alpha_1 P_{NBD}(N, <N>_1, k_1) + \alpha_2 P_{NBD}(N, <N>_2, k_2)$$

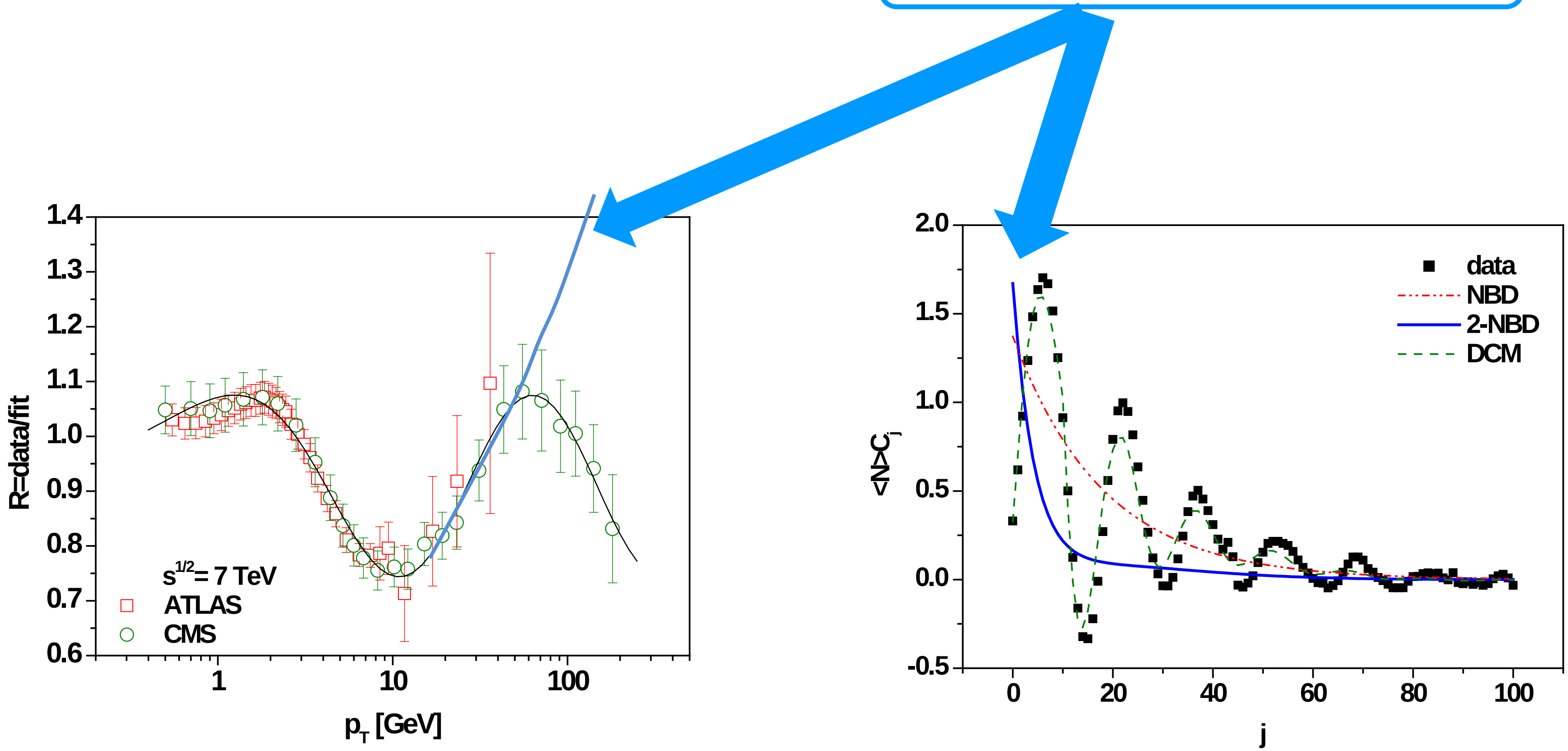



G.G.Barnafoldi et al., JPCS612(2015)012048

*A „soft+hard" model…"*

# Oscillation phenomena or two-component model




**Sensitivity of the ratio R and coefficients $C_j$ to the systematic uncertainties of the measurement and to the unfolding uncertainties can be checked only by the scrutinous analysis of the raw data with the proper response matrix (and that exceeds our capability).**

**However, in the case when these oscillations (or some other, equally nonexpected) would be experimentally confirmed, a new, fresh look at the dynamics of multiparticle production processes would be open.**

# Very recent result:

**It turns out that occurence of such oscillations do not eliminate the possible use of a multicomponent NBD. Namely, the multicomponent NBD can, after all, lead to the oscillatory behavior of coefficients $C_j$ .**

Let

$$P(N) = \sum_l \omega_l P_{NBD}(N, p_l)$$

be superposition of NBDs, $P_{NBD}(N, p_l)$, with weights $\omega_l$ and emission probability $p_l = m_l/(m_l + k_l)$. We can write

$$P(0)C_j = \frac{1}{<N>}(j+1)\sum_l \omega_l P_{NBD}(j+1, p_l) - \sum_{i=0}^{j-1} C_i \sum_l \omega_l P_{NBD}(j-i, p_l) =$$

$$(j+1)\left(\sum_l \omega_l \frac{P_{NBD}(j+1, p_l)}{<N>} - \sum_l \omega_l \frac{P_{NBD}(j+1, p_l)}{m_l}\right) + \sum_l \omega_l \left(\frac{1}{m_l}(j+1)P_{NBD}(j+1, p_l) - \sum_{i=0}^{j-1} C_i P_{NBD}(j-i, p_l)\right)$$

Using $C_j$ for NBD in the second summand of the above equation we have

$$P(0)C_j = (j+1)\left(\sum_l \omega_l P_{NBD}(j+1, p_l)\left(\frac{1}{<N>} - \frac{1}{m_l}\right)\right) + \sum_l \omega_l (1-p_l)^{k_l+1} p_l^j$$

and

$$P(0)C_j = \sum_l \omega_l p_l^j (1-p_l)^{k_l+1}\left\{\frac{\Gamma(j+k_l+1)}{\Gamma(k_l+1)\Gamma(j+1)}\frac{m_l - <N>}{<N>} + 1\right\}$$

For $m_l <$ <N> we have negative terms which can result in nonmonotonic behavior of coefficients $C_j$.

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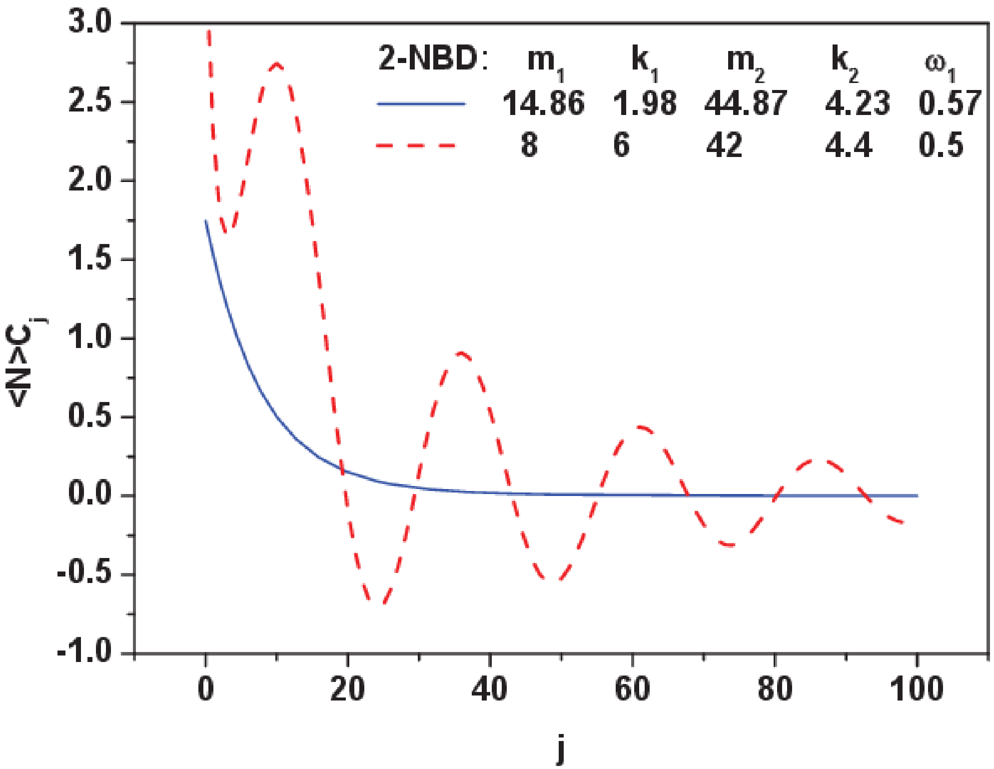
$$P(0)C_j = \sum_l \omega_l p_l^j (1-p_l)^{k_l+1} \left\{ \frac{\Gamma(j+k_l+1)}{\Gamma(k_l+1)\Gamma(j+1)} \frac{m_l - <N>}{<N>} + 1 \right\}$$

For $m_l <$ <N> we have negative terms which can result in nonmonotonic behavior of coefficients $C_j$.

# Very recent result:

**The 2-component NBD with suitably chosen parameters produces oscillations (dashed line). But those parameters are not the one used so far in fitting P(N) (full line).**



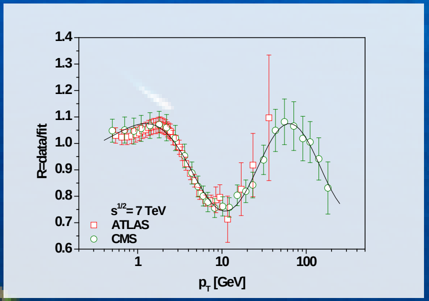

**Therefore:** ***Possible models of multiparticle production must describe, with the same parameters, both the*** ***multiplicity distributions P(N)*** ***and*** ***the corresponding coefficients $C_j$*** ***becsause these coefficients provide us a new information, which can be used to improve models of particle production processes.***

# 여러분의 관심 에 감사드립니다

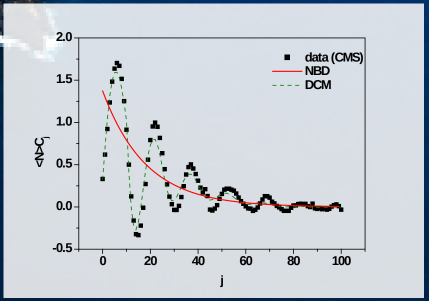
# Thank you for your attention

## and I look forward to your comments and questions

**References:**

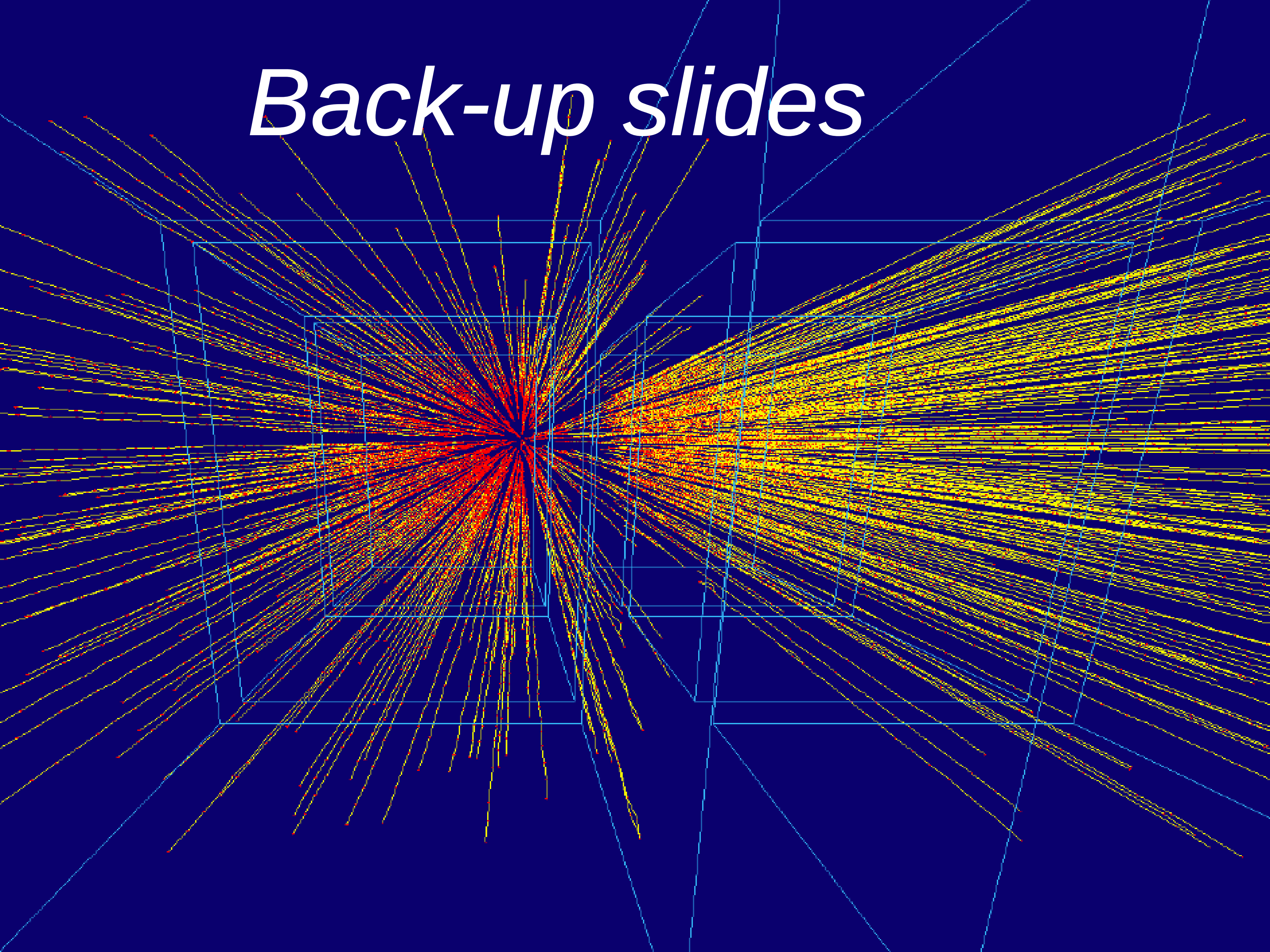



**Entropy 17 (2015) 384**
**Chaos, S & F 81 (2015) 487**




**arXiv:1601.03883**

# Back-up slides



## Frequency of oscillations

Comparison of fit parameters of oscillating term $R$ clearly show that observed frequency given by parameter $c$ is few tens smaller than expected

$$2\pi / \ln(1+\alpha)$$

value for any reasonable $\alpha$

## and the hierarchy of evolution

$$E_i = E_{i-1} + \alpha_{i-1}(nT + E_{i-1})$$

Neglecting the fluctuations of $\alpha_i$ parameters, after $\kappa$ steps

$$nT + E_\kappa = (1+\alpha)^\kappa (nT + E_0)$$

and

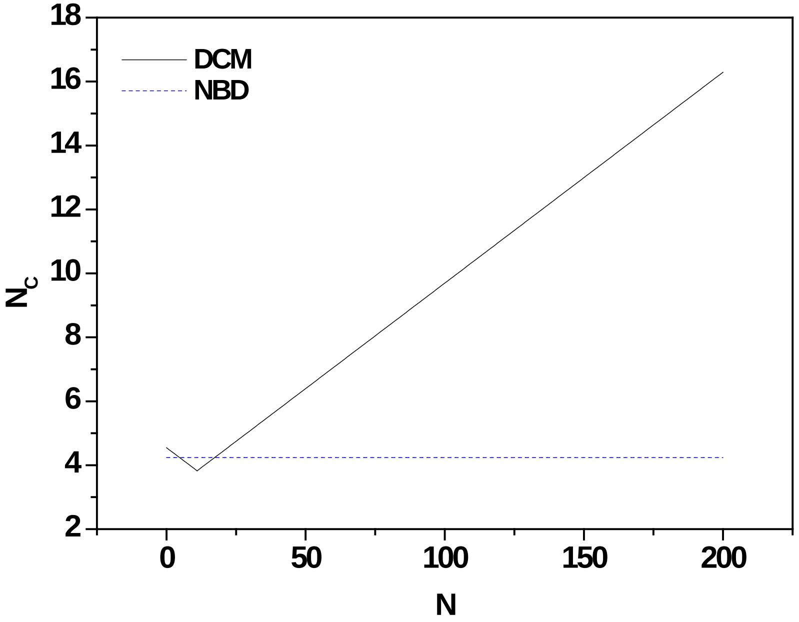
$$g\left((1+\alpha)^\kappa x\right) = (1-\alpha n)^\kappa g(x)$$

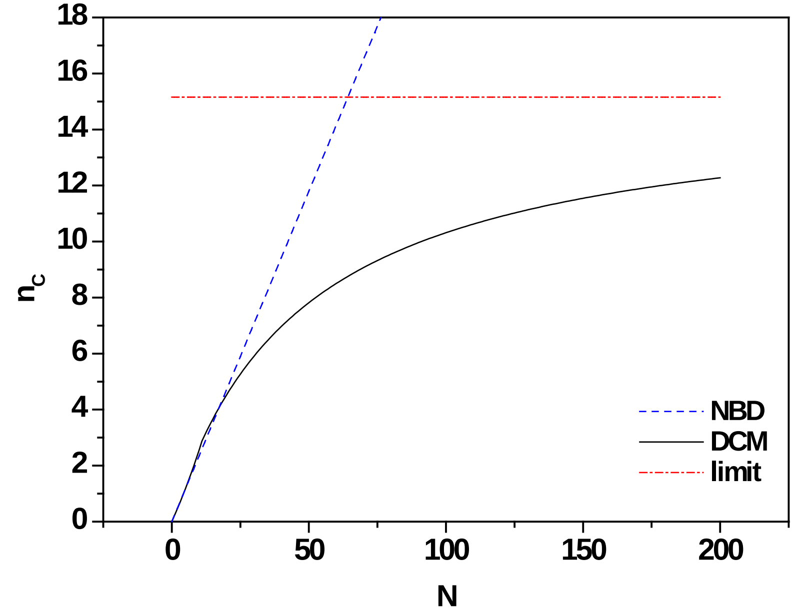
These equation do not change the slope parameter $m_0$ but frequency of oscillation

$$c = \frac{2\pi}{\kappa \ln(1+\alpha)}$$

becomes $\kappa$ times smaller

Experimental data indicate that $\kappa \cong 22$ (for $\alpha \cong 0.15$ and $c \cong 2$ )




$$n_C = (ak)^{-1}$$

number of clans

$$< N_C >= k \ln\left(1 + \frac{m}{k}\right)$$

$$N_C = N_C(N) \cong k \ln\left(\frac{m}{k}\right) = k\left[\ln\left(\frac{c}{k}\right) + a|N - b|\right]$$

multiplicity in a clan

$$n_c = \frac{N}{N_C} = \frac{N}{k\left[\ln\left(\frac{c}{k}\right) + a|N - b|\right]}$$

The NBD can also be defined by the following probability generating function

$$G_{NBD}(s) = \left(\frac{1-p}{1-ps}\right)^k \quad \text{where} \quad p = \frac{m}{m+k}. \tag{A.1}$$

Particles are registered with the probability $\alpha$ and their acceptance process is described by the binomial distribution with generating function

$$G_{BD}(s) = 1 - \alpha + \alpha s. \tag{A.2}$$

The number $N$ of registered particles is

$$N = \sum_{i=1}^{M} n_i. \tag{A.3}$$

where $n_i$ follows the BD and $M$ comes from the NBD. The generating function for the distribution of $N$ registered particles is then given by

$$G(s) = G_{BD}\left(G_{NBD}(s)\right) = 1 - \alpha + \alpha \left(\frac{1-p}{1-ps}\right)^k. \tag{A.4}$$

This corresponds to a probability distribution of registered particles

$$P(N) = \frac{1}{N!}\frac{d^N G(s)}{ds^N}\bigg|_{s=0}. \tag{A.5}$$

The corresponding recurrence relation for this distribution is

$$g(N) = \frac{(N+1)P(N+1)}{P(N)} = \frac{\frac{d^{N+1}G(s)}{ds^{N+1}}\Big|_{s=0}}{\frac{d^N G(s)}{ds^N}\Big|_{s=0}}. \tag{A.6}$$

Note that for $N > 0$ the function $g(N)$ does not depend on the acceptance and is the same as that for the NBD. However, for $N = 0$ the acceptance $\alpha$ enters and one has that

$$g(0) = \frac{\alpha\,(1-p)^k\,pk}{1-\alpha+\alpha\,(1-p)^k} = \frac{m\alpha\left(\frac{k}{m+k}\right)^{k+1}}{1-\alpha+\alpha\left(\frac{k}{m+k}\right)^k}. \tag{A.7}$$

In fact, the above result is valid for any distribution $P(M)$ with probability generating function $G_M(s)$, i.e., the term with $N = 0$,

$$g(0) = \frac{\alpha}{1-\alpha+\alpha G_M(0)}\frac{dG_M(s)}{ds}\bigg|_{s=0}, \tag{A.8}$$

always depends on the acceptance.